# On Wellposedness of Forward-Backward SDEs — A Unified Approach

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**Abstract**

In this paper we study the wellposedness of the forward-backward stochastic differential equations (FBSDE) in a general non-Markovian framework. The main purpose is to find a unified scheme which combines all existing methodology in the literature, and to overcome some fundamental difficulties that have been longstanding problems for non-Markovian FBSDEs. Our main devices are a *decoupling random field* and its associated *characteristic BSDE*, a backward stochastic Riccati-type equation with superlinear growth in both components $Y$ and $Z$. We establish various sufficient conditions under which the characteristic BSDE is wellposed, which leads to the existence of the decoupling random field, and ultimately to the solvability of the original FBSDE. We show that all existing frameworks could be analyzed using our new criteria.

**Keywords:** Forward-backward SDEs, decoupling random fields, characteristic BSDEs, backward stochastic Riccati equations, comparison theorem.

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# 1 Introduction

The theory of Backward Stochastic Differential Equations (BSDEs) and Forward-Backward Stochastic Differential Equations (FBSDEs) have been studied extensively for the past two decades, and its applications have been found in many branches of applied mathematics, especially the stochastic control theory and mathematical finance. It has been noted, however, that while in many situations the solvability of the original (applied) problems is essentially equivalent to the solvability of certain type of FBSDEs, these FBSDEs are often beyond the scope of any existing frameworks, especially when they are outside the Markovian paradigm, where the PDE tool becomes powerless. On the other hand, there have been some longstanding problems associated to FBSDEs that remain unsolved. Among others, the balance between the regularity of the coefficients and the time duration, as well as the non-degeneracy (of the forward diffusion), is still commonly recognized as the fundamental difficulty, and it is even more so in a general non-Markovian framework. Therefore it is becoming increasingly clear that the theory now calls for new insights and ideas that can lead to a better understanding of the problem and hopefully to a unified solution scheme for the general FBSDEs.

A strongly coupled FBSDE takes the following form:

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s, Z_s)dB_s; \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \end{cases} \qquad t \in [0, T], \tag{1.1}$$

where $b$, $f$, and $\sigma$ are (progressively) measurable functions defined on appropriate spaces, $B$ is a standard Brownian motion, and $g$ is a (possibly random) function that is defined on $\mathbb{R}^n \times \Omega$ such that $g(x, \cdot)$ is $\mathcal{F}_T$-measurable for each fixed $x$.

There have been three main methods to solve FBSDE (1.1). First, the *Method of Contraction Mapping.* This method, first used by Antonelli [1] and later detailed by Pardoux-Tang [16], works well when the duration $T$ is relatively small. Second, the *Four Step Scheme.* This was the first solution method that removed restriction on the time duration for Markovian FBSDEs, initiated by Ma-Protter-Yong [11]. The trade-off is the requirement on the regularity of the coefficients so that a "decoupling" quasi-linear PDE has a classical solution. Third, the *Method of Continuation.* This was a method that can treat non-Markovian FBSDEs with arbitrary duration, initiated by Hu-Peng [6] and Peng-Wu [17], and later developed by Yong [23] and recently in [26]. The main assumption for this method is the so-called "monotonicity conditions" on the coefficients, which is restrictive in a different way. This method has been used widely in applications (see, e.g., [20, 27, 22]) because of its pure probabilistic nature. We refer to the book Ma-Yong [15] for the

detailed accounts for all three methods. It is worth noting that these three methods do not cover each other.

In this paper we shall carry out a systematic analysis for a decoupling scheme, and develop a strategy to construct a decoupling random field which will be the key to the solvability of general non-Markovian FBSDEs. Our starting point is the work of Delarue [4], in a Markovian framework with $\sigma = \sigma(t, x, y)$ being uniformly nondegenerate. In that case an FBSDE over arbitrary time duration was solved under only Lipschitz conditions on the coefficients, by combining nicely the method of contraction mapping and the Four Step Scheme and some delicate PDE arguments. The idea was later extended by Zhang [28] to the non-Markovian cases (again in the case $\sigma = \sigma(t, x, y)$), by using mainly probabilistic arguments, and with the help of some compatibility conditions. The main point is still, as in the Four Step Scheme, around finding a function $u$ such that

$$Y_t = u(t, X_t), \qquad t \in [0, T]. \tag{1.2}$$

Clearly, if the FBSDE (1.1) is non-Markovian, then $u$ should be a random field. The key issue here, as we shall argue, is the existence of such a *decoupling* random field that is uniformly Lipschitz in its spatial variable. We will show that the existence of such a random field is closely related to the solvability of an associated BSDE (called the *characteristic BSDE* in this paper), and will ultimately lead to the wellposedness of the original FBSDEs. We shall provide a set of sufficient conditions for the existence of such decoupling field, and show that all existing frameworks in the literature could be analyzed by using our criteria. Furthermore, we note that in the case when the FBSDE is linear with constant coefficients, some of our conditions are actually necessary. In other words, these conditions *cannot* be improved.

A brief description of our plan is as follows. Assume that the decoupling field $u$ exists and the FBSDE is wellposed. Denote $(X^x, Y^x, Z^x)$ to be the solution to FBSDE (1.1) with initial value $x$. Then we argue that the derivative of $(X^x, Y^x, Z^x)$ with respect to $x$, denoted by $(\nabla X, \nabla Y, \nabla Z)$, would satisfy the following linear "variational FBSDE":

$$\begin{cases} \nabla X_t = 1 + \displaystyle\int_0^t (b_1 \nabla X_s + b_2 \nabla Y_s + b_3 \nabla Z_s) ds + \int_0^t (\sigma_1 \nabla X_s + \sigma_2 \nabla Y_s + \sigma_3 \nabla Z_s) dB_s; \\ \nabla Y_t = h \nabla X_T + \displaystyle\int_t^T (f_1 \nabla X_s + f_2 \nabla Y_s + f_3 \nabla Z_s) ds - \int_t^T \nabla Z_s dB_s, \qquad t \in [0, T], \end{cases} \tag{1.3}$$

where the coefficients are bounded processes. Since $Y_t^x = u(t, X_t^x)$ by (1.2), we must have $\nabla Y_t = u_x(t, X_t) \nabla X_t$, and thus $u_x(t, X_t) = \nabla Y_t (\nabla X_t)^{-1} \stackrel{\triangle}{=} \hat{Y}_t$. So proving $u$ is uniformly Lipschitz continuous is essentially equivalent to finding solutions to the linear FBSDE (1.3) such that $\hat{Y} = \{\hat{Y}_t\}$ is uniformly bounded. Furthermore, one can then check that $\hat{Y}$ actually satisfies a BSDE (see (3.6) below) which will be called the *characteristic BSDE* in this paper. We note that

this BSDE has super-linear growth in both components of the solutions, thus it is itself a novel subject in BSDE theory, thus is interesting in its own right.

Our main task for analyzing the characteristic BSDE is to show that it actually possesses a uniformly bounded solution $\hat{Y}$ over an arbitrary time duration $[0, T]$. We shall accomplish this by studying two dominating ODEs (see (3.12) below), which bound $\hat{Y}$ from above and below, respectively. While it is not difficult to obtain the local existence of the solutions to these dominating ODEs, finding the global solution of ODE (3.12) over arbitrary duration $[0, T]$ is by no means trivial, due to the combined complexity from its nonlinearity, super-linear growth, and the singularity. We shall give a set of mild sufficient conditions to guarantee the existence of the solutions to the ODEs, which in turn guarantees the solvability of the original FBSDE (1.1). Our results extend those of [28] in many ways, and we believe they are by far the most general criteria for the solvability of FBSDEs. As a byproduct, we also prove a comparison theorem for the decoupling random field over all time, thus confirming a common belief (see, e.g., [15, 19, 21]).

There are several technical aspects in this paper that are worth emphasizing. First, the linear FBSDE was studied by Yong [24, 25], and the associated characteristic BSDE was also observed in [25]. But since in that work the main focus was to treat the quadratic growth of $\hat{Z}$, conditions were made so that the generator is linear in $\hat{Y}$, which reduced the complexity drastically. On the other hand, our characteristic BSDE extends the so-called backward stochastic Riccati equation, often seen in the Linear-Quadratic stochastic control literature (see, e.g., [10] and [18]). But to our best knowledge, to date there has been no result on BSDEs with at least quadratic growth on both components. Second, the decoupling random field has also been found through quasilinear PDEs and backward SPDEs (see, e.g., [4, 5, 11, 13, 14]), but our method requires the minimum assumptions on the coefficients, and covers both Markovian and non-Markovian cases. In an accompanying paper [12], we show that the FBSDE has a uniformly Lipschitz continuous decoupling field (and thus is wellposed) if and only if the corresponding quasi-linear BSPDE has a uniformly Lipschitz continuous Sobolev type weak solution. We hope that this connection can enhance further understanding on both FBSDEs and BSPDEs. Third, the method in this paper is particularly effective for the cases where the forward diffusion coefficient $\sigma$ depends on $Z$, which has been avoided in many existing works, as it brings in some extra complications for the solvability analysis (see, e.g., [4, 15]). Finally, in this paper we content ourselves for one dimensional FBSDEs. In fact, the characteristic BSDE becomes much more subtle in high dimensional cases, as it involves the combination of high dimensional BSDEs with quadratic growth (in $Z$) and high dimensional backward stochastic Riccati equations, each of which is very challenging. We hope to be able to address this issue in our future publications.

The rest of the paper is organized as follows. In Section 2 we introduce the decoupling field

and show how it leads to the wellposedness of FBSDEs. In Section 3 we heuristically discuss our strategy for obtaining the uniformly Lipschitz continuity of the decoupling field. In Section 4 we study the relation between the solvability of the linear variational FBSDE and its characteristic BSDE, and in Section 5 we investigate the global solutions of the dominant ODEs. In section 6 we investigate the wellposedness of FBSDEs over small time duration, and in Section 7 we conclude our wellposedness result for general FBSDEs over arbitrary time interval. In Section 8 we prove several further properties of FBSDEs. Finally in Appendix we complete some technical proofs.

## 2 The Decoupling Field

Throughout this paper we denote $(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{F})$ to be a filtered probability space on which is defined a Brownian motion $B = (B_t)_{t\geq 0}$. We assume that $\mathbb{F} \stackrel{\triangle}{=} \mathbb{F}^B \stackrel{\triangle}{=} \{\mathcal{F}^B_t\}_{t\geq 0}$, the natural filtration generated by $B$, augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$. For any sub-$\sigma$-filed $\mathcal{G} \subseteq \mathcal{F}$, and $0 \leq p \leq \infty$, we denote $L^p(\mathcal{G})$ to be the spaces of all $\mathcal{G}$-measurable, $L^p$-integrable random variables. In what follows we assume that *all processes involved are 1-dimensional.*

Let $T > 0$ be a fixed time horizon. We consider the general FBSDEs (1.1) for $t \in [0, T]$, where the coefficients $b, \sigma, f, g$ are measurable functions, and are allowed to be random in general. For technical clarity, we shall make use of the following *Standing Assumptions* throughout this paper:

**Assumption 2.1** *(i) The coefficients $b, \sigma, f : [0, T] \times \Omega \times \mathbb{R}^3 \mapsto \mathbb{R}$ are $\mathbb{F}$-progressively measurable, for fixed $(x, y, z) \in \mathbb{R}^3$; and the function $g : \mathbb{R} \times \Omega \mapsto \mathbb{R}$ is $\mathcal{F}_T$-measurable, for fixed $x \in \mathbb{R}$. Moreover, the following integrability condition holds:*

$$I_0^2 \stackrel{\triangle}{=} \mathbb{E}\Big\{\big(\int_0^T [|b| + |f|]^2(t, 0, 0, 0)dt\big)^2 + \int_0^T |\sigma|^2(t, 0, 0, 0)dt + |g(0)|^2\Big\} < \infty. \tag{2.1}$$

*(ii) The coefficients $b, \sigma, f, g$ are uniformly Lipschitz continuous in the spatial variable $(x, y, z) \in \mathbb{R}^3$, uniformly in $\omega \in \Omega$, and with a common Lipschitz constant $K_0 > 0$.* ■

To simplify notations, throughout the paper we denote $\Theta \stackrel{\triangle}{=} (X, Y, Z)$. Our purpose is to find $\mathbb{F}$-progressively measurable, square-integrable processes $\Theta$, such that (1.1) holds for all $t \in [0, T]$, $\mathbb{P}$-a.s. However, to facilitate the discussion, in what follows we often consider the FBSDE on a subinterval $[t, T]$, so that $X_t = x$, $\mathbb{P}$-a.s., and we denote the corresponding solution, if exists, by $\Theta^{t,x}_s$, $s \in [t, T]$, as usual. Moreover, for $0 \leq t_1 < t_2 \leq T$, we also consider the following FBSDE

on $[t_1, t_2]$:

$$\begin{cases} X_t = \eta + \int_{t_1}^t b(s, \Theta_s)ds + \int_{t_1}^t \sigma(s, \Theta_s)dB_s; \\ Y_t = \varphi(X_{t_2}) + \int_t^{t_2} f(s, \Theta_s)ds - \int_t^{t_2} Z_s dB_s, \end{cases} \qquad t \in [t_1, t_2], \tag{2.2}$$

where $\eta \in L^2(\mathcal{F}_{t_1})$ and $\varphi(x, \cdot) \in L^2(\mathcal{F}_{t_2})$, for each fixed $x$. We denote the solution to FBSDE (2.2), if exists, by $\Theta^{t_1,t_2,\eta,\varphi}$.

A well understood technique for solving an FBSDE, initiated in [11], is to find a "decoupling function" $u$ so that the solution $\Theta$ to the FBSDE satisfies the relation (1.2). In Markovian cases, especially when $\sigma = \sigma(t, x, y)$, it was shown that $u$ is related to the solution to a quasilinear PDE, either in classical sense or in viscosity sense (cf. e.g., [11], [4], or [16]). When the coefficients are allowed to be random, special cases were also studied and the function $u$ was found either as the solution to certain backward stochastic PDEs (see, [13, 14]), or as a random field constructed by extending the localization technique of [4] under certain compatibility conditions of the coefficients (see, [28]). In the sequel we call such random function $u$ the *decoupling random field* or simply *decoupling field* of the FBSDE (1.1). More precisely we have the following definition.

**Definition 2.2** *A (random) function $u : [0, T] \times \mathbb{R} \times \Omega \mapsto \mathbb{R}$ is said to be a "decoupling field" of FBSDE (1.1) if:*

*(i) $u(T, x) = g(x)$, $\mathbb{P}$-a.s., for all $x \in \mathbb{R}$;*

*(ii) $u$ is $\mathbb{F}$-progressively measurable for each $x \in \mathbb{R}$, and is uniformly Lipschitz continuous in $x$ with a Lipschitz constant $K > 0$; and*

*(iii) There exists a constant $\delta \stackrel{\triangle}{=} \delta(K_0, K) > 0$, where $K_0$ is the Lipschitz constant in Assumption 2.1, such that for any $0 = t_1 < t_2 \le T$ with $t_2 - t_1 \le \delta$, and any $\eta \in L^2(\mathcal{F}_{t_1})$, the FBSDE (2.2) with initial value $\eta$ and terminal condition $u(t_2, \cdot)$ has a unique solution that satisfies (1.2) for $t \in [t_1, t_2]$, $\mathbb{P}$-almost surely.* ■

By a slight abuse of notation, we shall denote the solution in part (iii) of Definition 2.2 by $\Theta^{t_1,t_2,\eta,u}$. One should note that the existence of the decoupling field implies that the well-posedness of the FBSDE over a small interval, which is usually guaranteed by the method of contraction mapping given the Assumption 2.1. The following result shows the significance of the existence of the decoupling field for the well-posedness for FBSDEs over an arbitrary duration.

**Theorem 2.3** *Assume that Assumption 2.1 holds, and that there exists a decoupling field $u$ for FBSDE (1.1). Then, FBSDE* (1.1) *has a unique solution $\Theta$ and (1.2) holds over an arbitrary duration* $[0, T]$.

*Proof.* Let $T > 0$ be given. Consider a partition: $0 = t_0 < \cdots < t_n = T$ of $[0, T]$ such that $t_{i+1} - t_i \le \delta$, $i = 0, \cdots, n-1$, where $\delta$ is the constant in Definition 2.2 (iii).

Define $X_{t_0} \stackrel{\triangle}{=} x$, $Y_{t_0} \stackrel{\triangle}{=} u(0, x)$, and for $i = 0, \cdots, n-1$, define recursively

$$\Theta_t \stackrel{\triangle}{=} \Theta_t^{t_i, t_{i+1}, X_{t_i}, u}, \quad t \in (t_i, t_{i+1}].$$

Then $\Theta$ would solve FBSDE (1.1) if they could be "patched" together. But note that

$$\begin{aligned} X_{t_i+} &= X_{t_i+}^{t_i, t_{i+1}, X_{t_i}, u} = X_{t_i}^{t_i, t_{i+1}, X_{t_i}, u} = X_{t_i}; \\ Y_{t_i+} &= Y_{t_i+}^{t_i, t_{i+1}, X_{t_i}, u} = Y_{t_i}^{t_i, t_{i+1}, X_{t_i}, u} = u(t_i, X_{t_i}) = Y_{t_i}^{t_{i-1}, t_i, X_{t_{i-1}}, u} = Y_{t_i}. \end{aligned}$$

That is, $(X, Y)$ is continuous on $[0, T]$. Moreover, $u(T, x) = g(x)$, then by (2.2) one can check straightforwardly that $\Theta$ satisfies FBSDE (1.1) on $[0, T]$, proving the existence. Furthermore, from our construction it is clear that (1.2) holds.

We now prove the uniqueness. Let $\tilde{\Theta}$ be an arbitrary solution to FBSDE (1.1). Note that $\tilde{\Theta}$ satisfies FBSDE (2.2) on $[t_{n-1}, t_n]$ with initial condition $\tilde{X}_{t_{n-1}}$. Then by the definition of the decoupling field we have $\tilde{Y}_{t_{n-1}} = u(t_{n-1}, \tilde{X}_{t_{n-1}})$. This implies that $\tilde{\Theta}$ satisfies FBSDE (2.2) on $[t_{n-2}, t_{n-1}]$ with initial condition $\tilde{X}_{t_{n-2}}$. Then we have $\tilde{Y}_{t_{n-2}} = u(t_{n-2}, \tilde{X}_{t_{n-2}})$. Repeating the arguments backwardly in time we obtain that $\tilde{Y}_{t_i} = u(t_i, \tilde{X}_{t_i})$, $i = n, \cdots, 0$. Now consider FBSDE (2.2) on $[t_0, t_1]$. Since $\tilde{X}_{t_0} = x = X_{t_0}$, by the uniqueness of solutions we know that $\tilde{\Theta} = \Theta$ on $[t_0, t_1]$. In particular, $\tilde{X}_{t_1} = X_{t_1}$, and thus the corresponding FBSDEs (2.2) on $[t_1, t_2]$ have the same initial condition. Repeating the arguments, this time forwardly for $i = 1, \cdots, n$, we see that $\tilde{\Theta} = \Theta$ on $[0, T]$ and thus the solution is unique. ■

We conclude this section by making the following observations.

**Remark 2.4** (i) Definition 2.2 and Theorem 2.3 can be extended to higher dimensional cases (with the constant $\delta$ possibly depending on the dimensions as well), and the proof stays exactly the same.

(ii) By the uniqueness in Theorem 2.3, it is obvious that the decoupling field, if exists, is also unique. In fact, it is clear that $u(t, x) = Y_t^{t,x}$.

(iii) As we will see later in this paper, the uniform Lipschitz continuity of $u$ is crucial for the wellposedness of the FBSDE. Such a feature was also observed from a different angle in [12], in which we characterize the decoupling field $u$ as a Sobolev type weak solution to certain *backward stochastic PDE*. In particular, in that work a decoupling field $u$ is called *regular* if it is uniform Lipschitz continuous in its spatial variable. We note that the idea of "decoupling device" was also used for linear FBSDEs in [25]. But in that work the uniform Lipschitz continuity was not studied. ■

## 3 Some Heuristic Analysis

From Theorem 2.3 it is easy to see that the issue of the wellposedness of FBSDE (1.1) can be decomposed into two parts. First, the wellposedness on small time interval, and second, finding a decoupling field $u$ that is uniformly Lipschitz continuous in its spatial variables. The first issue was more or less "classical" (see, e.g., [1]), but we will fine-tune it in Section 6 to suit our purpose. The second issue, however, is much more subtle, and is the main focus of this paper. In this section we first give a heuristic analysis, from which several fundamental problems will be formulated, and their proofs will be carried out in Sections 4 and 5 below. A synthetic analysis will then be given in Section 7.

Our main idea to decouple the FBSDE (1.1) is as follows. Assume that there exists a decoupling field $u = u(t,x)$ that is uniformly Lipschitz continuous in $x$ and (1.2) holds. Assume also that (1.1) is well-posed on $[0,T]$, with $X_0 = x$ for any $x$. Given $x_1 \neq x_2$, let $\Theta^i$, $i = 1,2$, denote the unique solution to (1.1) with initial condition $x_i$. Let us denote

$$\nabla\Theta \triangleq \frac{\Theta^1 - \Theta^2}{x_1 - x_2}, \qquad \nabla u(t) \triangleq \frac{u(t,X_t^1) - u(t,X_t^2)}{X_t^1 - X_t^2}. \tag{3.1}$$

Since $Y_t^i = u(t,X_t^i)$, $i = 1,2$, one must have

$$\nabla Y_t = \nabla u(t)\nabla X_t, \tag{3.2}$$

and one can check immediately that $\nabla\Theta$ satisfies the linear "variational FBSDE" (1.3) in which $h \triangleq \frac{g(X_T^1)-g(X_T^2)}{X_T^1-X_T^2}\mathbf{1}_{\{X_T^1\neq X_T^2\}}$; and, for $\varphi = b,\sigma,f$, respectively,

$$\begin{aligned}
\varphi_1(t) &\triangleq \frac{\varphi(t,X_t^1,Y_t^1,Z_t^1) - \varphi(t,X_t^2,Y_t^1,Z_t^1)}{X_t^1 - X_t^2}\mathbf{1}_{\{X_t^1\neq X_t^2\}};\\
\varphi_2(t) &\triangleq \frac{\varphi(t,X_t^2,Y_t^1,Z_t^1) - \varphi(t,X_t^2,Y_t^2,Z_t^1)}{Y_t^1 - Y_t^2}\mathbf{1}_{\{Y_t^1\neq Y_t^2\}};\\
\varphi_3(t) &\triangleq \frac{\varphi(t,X_t^2,Y_t^2,Z_t^1) - \varphi(t,X_t^2,Y_t^2,Z_t^2)}{Z_t^1 - Z_t^2}\mathbf{1}_{\{Z_t^1\neq Z_t^2\}}.
\end{aligned} \tag{3.3}$$

We note here that $b_i,\sigma_i,f_i$, $i = 1,2,3$, are $\mathbb{F}$-adapted processes and $h$ is a $\mathcal{F}_T$-measurable random variable, and they are all bounded, thanks to Assumption 2.1.

Furthermore, in light of (3.2) we see that $u$ being uniformly Lipschitz continuous in $x$ is essentially equivalent to that $\hat{Y}_t \triangleq \nabla Y(\nabla X)^{-1}$ being uniformly bounded. Thus, let us assume $\nabla X \neq 0$ and denote

$$\hat{Y}_t \triangleq \nabla Y_t/\nabla X_t \quad \text{and} \quad \hat{Z}_t \triangleq [\nabla Z_t - \hat{Y}_t(\sigma_1\nabla X_t + \sigma_2\nabla Y_t + \sigma_3\nabla Z_t)]/\nabla X_t, \tag{3.4}$$

or equivalently,

$$\nabla Y_t = \hat{Y}_t \nabla X_t, \quad \nabla Z_t = \frac{\hat{Z}_t + \hat{Y}_t(\sigma_1 + \sigma_2 \hat{Y}_t)}{1 - \sigma_3 \hat{Y}_t} \nabla X_t. \tag{3.5}$$

A simple application of Itô's formula to $\hat{Y}_t$, assuming $\sigma_3 \hat{Y} \neq 1$, yields that

$$\hat{Y}_t = h + \int_t^T \Big[ F_s(\hat{Y}_s) + G_s(\hat{Y}_s)\hat{Z}_s + \Lambda_s(\hat{Y}_s)|\hat{Z}_s|^2 \Big] ds - \int_t^T \hat{Z}_s dB_s, \tag{3.6}$$

where

$$\begin{aligned} F_s(y) &\triangleq f_1 + f_2 y + y(b_1 + b_2 y) + \frac{(f_3 + b_3 y) y (\sigma_1 + \sigma_2 y)}{1 - \sigma_3 y}; \\ G_s(y) &\triangleq \sigma_1 + \sigma_2 y + \frac{f_3 + b_3 y + \sigma_3 y(\sigma_1 + \sigma_2 y)}{1 - \sigma_3 y} = \frac{(\sigma_1 + f_3) + (\sigma_2 + b_3) y}{1 - \sigma_3 y}; \\ \Lambda_s(y) &\triangleq \frac{\sigma_3}{1 - \sigma_3 y}. \end{aligned} \tag{3.7}$$

The equation (3.6) is clearly a legitimate BSDE, even without assuming $\nabla X \neq 0$. We shall call this BSDE the "*Characteristic BSDE*" of the linear variational FBSDE (1.3), and their connection will be studied rigorously in the next section. In fact, by virtue of Theorem 2.3, and the relation between the variational FBSDE and the uniform Lipschitz decoupling field $u$, we often refer to (3.6) as the characteristic BSDE for the original FBSDE (1.1). Our main task in the rest of the paper is to find conditions so that the BSDE (3.6) has a solution $(\hat{Y}, \hat{Z})$ such that

$$\text{both } \hat{Y} \text{ and } (1 - \sigma_3 \hat{Y})^{-1} \text{ are bounded.} \tag{3.8}$$

**Remark 3.1** It is worth noting that the BSDE (3.6) is nonstandard in several aspects. Most notable is that its generator has at least quadratic growth in both $Y$ and $Z$, thus it can be thought of as a Backward Stochastic Riccati Equations (BSRE) with quadratic growth in $Z$, which, to our best knowledge, has not been studied in literature.

Besides the commonly cited reference of BSDEs with quadratic growth in $Z$ (e.g., [9], [2]), the following special cases of (3.6) are worth mentioning. In [18] the BSRE with linear growth in $Z$ was studied in the context of stochastic LQ (linear-quadratic) problem, in which the FBSDE is a natural consequence of the stochastic maximum principle. The characteristic BSDE (3.6) was also observed in [25], where the linear FBSDEs were considered. But some special assumptions were made so that the BSDE has linear growth in $Y$. Finally, in [28] certain compatibility conditions were also added so that (3.6) becomes a standard BSDE and thus its wellposedness was not an issue. Our results will contain those of [18], [25], and [28] as special cases. ■

We conclude this section by outlining the strategy for obtaining the *a priori* uniform estimate of $\hat{Y}$, which is crucial for finding the solution of (3.6) satisfying (3.8). To begin with, for any bounded random variable $\xi$, define its deterministic upper and lower bounds by

$$\overline{\xi} \stackrel{\triangle}{=} \operatorname{esssup} \xi \stackrel{\triangle}{=} \inf\{a \in \mathbb{R} : \xi \leq a, \text{ a.s.}\}, \quad \underline{\xi} \stackrel{\triangle}{=} \operatorname{essinf} \xi \stackrel{\triangle}{=} \sup\{a \in \mathbb{R} : \xi \geq a, \text{ a.s.}\} \tag{3.9}$$

Next, let $g$ be the terminal function in (1.1) and recall the function $F(t, y)$ defined by (3.7). For any $x_1 \neq x_2$, $y_1 \neq y_2$, $z_1 \neq z_2$, we denote

$$h(x_1, x_2) \stackrel{\triangle}{=} \frac{g(x_1) - g(x_2)}{x_1 - x_2}. \tag{3.10}$$

Further, we first replace the coefficients $b_i, \sigma_i, f_i$, $i = 1, 2, 3$, in (3.7) by those in (3.3), then replace $\Theta^j$ in (3.3) by $\theta_j \stackrel{\triangle}{=} (x_j, y_j, z_j)$, $j = 1, 2$, and denote the resulting function by $F(\theta_1, \theta_2; t, y)$. We then define

$$\begin{aligned}
&\overline{h} \stackrel{\triangle}{=} \operatorname{esssup}\Big(\sup_{x_1 \neq x_2} h(x_1, x_2)\Big), \quad \underline{h} \stackrel{\triangle}{=} \operatorname{essinf}\Big(\inf_{x_1 \neq x_2} h(x_1, x_2)\Big); \\
&\overline{F}(t, y) \stackrel{\triangle}{=} \operatorname{esssup}\Big(\sup_{x_1 \neq x_2, y_1 \neq y_2, z_1 \neq z_2} F(\theta_1, \theta_2; t, y)\Big), \\
&\underline{F}(t, y) \stackrel{\triangle}{=} \operatorname{essinf}\Big(\inf_{x_1 \neq x_2, y_1 \neq y_2, z_1 \neq z_2} F(\theta_1, \theta_2; t, y)\Big).
\end{aligned} \tag{3.11}$$

Here we should remark that $\overline{F}(t, y)$ is a *deterministic* function, and we should note its notational difference from the possibly random processes, e.g. $F_t(y)$, $G_t(y)$, etc, appeared previously. We have the following a priori estimate of $\hat{Y}$.

**Lemma 3.2** *Let Assumption 2.1 hold, and the* $\underline{h}$, $\overline{h}$, $\underline{F}(t, y)$, $\overline{F}(t, y)$ *defined in* (3.11) *satisfy* $\underline{h} \leq h \leq \overline{h}$, $\underline{F}(t, \hat{Y}_t) \leq F(t, \hat{Y}_t) \leq \overline{F}(t, \hat{Y}_t)$. *Assume that the BSDE (3.6) has a solution* $(\hat{Y}, \hat{Z})$, *and the following ordinary differential equations (ODEs) admit solutions* $\overline{\mathbf{y}}, \underline{\mathbf{y}}$*:*

$$\overline{\mathbf{y}}_t = \overline{h} + \int_t^T \overline{F}(s, \overline{\mathbf{y}}_s) ds, \quad \underline{\mathbf{y}}_t = \underline{h} + \int_t^T \underline{F}(s, \underline{\mathbf{y}}_s) ds. \tag{3.12}$$

*Assume further that* $\hat{Y}$, $\overline{\mathbf{y}}$, *and* $\underline{\mathbf{y}}$ *all satisfy (3.8). Then* $\underline{\mathbf{y}}_t \leq \hat{Y}_t \leq \overline{\mathbf{y}}_t$, *for all* $t \in [0, T]$, $\mathbb{P}$*-a.s.*

*Proof.* Denote $\tilde{G}_t(z) \stackrel{\triangle}{=} G_t(\hat{Y}_t)z + \Lambda_t(\hat{Y}_t)z^2$. Note that $(\hat{Y}, \hat{Z})$ satisfies the following BSDE:

$$Y_t = h + \int_t^T [F_s(Y_s) + \tilde{G}_s(Z_s)] ds - \int_t^T Z_s dB_s$$

and $(\overline{\mathbf{y}}, 0)$ satisfy the following BSDE:

$$Y_t = \overline{h} + \int_t^T [\overline{F}(s, Y_s) + \tilde{G}_s(Z_s)] ds - \int_t^T Z_s dB_s.$$

Let $C > 0$ be the common upbound of $|\hat{Y}|$, $|1 - \sigma_3 \hat{Y}|^{-1}$, $|\overline{\mathbf{y}}|$, $|1 - \overline{\mathbf{y}}|^{-1}$, $|\underline{\mathbf{y}}|$, and $|1 - \underline{\mathbf{y}}|^{-1}$. Note that $F$ is uniformly Lipschitz continuous in $y$ in the set $\{y : |y| \le C, |1 - \sigma_3 y|^{-1} \le C\}$. It then follows from the comparison theorem for quadratic BSDEs (see, e.g., [9]) that $\hat{Y} \le \overline{\mathbf{y}}$. Similarly we have $\hat{Y} \ge \underline{\mathbf{y}}$. ■

Combining the discussions in sections 2 and 3, especially Lemma 3.2, it is now clear that finding the uniform Lipschitz decoupling random field $u$ will eventually come down to finding conditions so that the ODEs in (3.12) admit non-explosive solutions over the arbitrarily prescribed duration $[0, T]$. In the rest of the paper we shall call the ODEs in (3.12) the "*dominating ODE*" of BSDE (3.6), whose well-posedness will be the main subject of Section 5.

# 4 The Characteristic BSDE

In this section we study the connection between well-posedness of the linear variational FBSDE (1.3) and the corresponding characteristic BSDE (3.6). We note that the variational FBSDE coincides with the original FBSDE if (1.1) is linear. For notational simplicity, we assume in this section that the FBSDE (1.1) is linear with random coefficients, and bearing in mind that the result in this section applies to the general variational FBSDEs.

$$\begin{cases} X_t = 1 + \int_0^t (b_1 X_s + b_2 Y_s + b_3 Z_s) ds + \int_0^t (\sigma_1 X_s + \sigma_2 Y_s + \sigma_3 Z_s) dB_s; \\ Y_t = h X_T + \int_t^T (f_1 X_s + f_2 Y_s + f_3 Z_s) ds - \int_t^T Z_s dB_s. \end{cases} \tag{4.1}$$

In this case (3.4) and (3.5) become

$$\hat{Y} \triangleq Y/X, \quad \hat{Z} \triangleq [Z - \hat{Y}(\sigma_1 X + \sigma_2 Y + \sigma_3 Z)]/X, \tag{4.2}$$

$$Y = \hat{Y} X, \quad Z = \Big[\hat{Z} + \hat{Y}(\sigma_1 + \sigma_2 \hat{Y})\Big] X / [1 - \sigma_3 \hat{Y}]. \tag{4.3}$$

The original Assumption 2.1 can be translated into the following:

**Assumption 4.1** *Assume $b_i, \sigma_i, f_i$, $i = 1, 2, 3$, are $\mathbb{F}$-adapted processes, $h$ is a $\mathcal{F}_T$-measurable random variable, and they are all bounded.*

The following spaces are important in our discussion. For $p \ge 1$, denote

$$\begin{aligned} \mathbb{L}^p &\triangleq \Big\{\Theta : \|\Theta\|_{\mathbb{L}^p}^p := \mathbb{E}\Big\{\sup_{0 \le t \le T} [|X_t|^p + |Y_t|^p] + \Big(\int_0^T |Z_t|^2 dt\Big)^{\frac{p}{2}}\Big\} < \infty\Big\}; \\ \widehat{\mathbb{L}}_p &\triangleq \bigcup_{q > p} \mathbb{L}^q. \end{aligned} \tag{4.4}$$

We begin our discussion with the following observation. For any $\mathbb{F}$-adapted process $u$ such that $\int_0^T |u_t|^2dt < \infty$, $\mathbb{P}$-a.s., we define

$$M_t^u \triangleq \exp\Big\{\int_0^t u_s dB_s - \frac{1}{2}\int_0^t |u_s|^2 ds\Big\}. \tag{4.5}$$

Consider the following simplified form of (3.6):

$$\hat{Y}_t = h + \int_t^T [\alpha_s + \beta_s \hat{Y}_s + \gamma_s \hat{Z}_s + \lambda_s |\hat{Z}_s|^2] ds - \int_t^T \hat{Z}_s dB_s, \quad t \in [0,T], \tag{4.6}$$

where $\alpha, \beta, \gamma, \lambda$ are $\mathbb{F}$-adapted processes and $h$ is an $\mathcal{F}_T$-measurable random variable, all bounded. Then it is well-known (see, e.g., [2]) that the BSDE (4.6) admits a unique solution $(\hat{Y}, \hat{Z})$ such that $\hat{Y}$ is bounded and the following estimates hold:

$$\mathbb{E}_t\Big\{\int_t^T |\hat{Z}_s|^2 ds\Big\} \le C \quad \text{and} \quad \mathbb{E}_t\Big\{\int_t^T |\lambda_t \hat{Z}_s|^2 ds\Big\} \le C.$$

Furthermore, applying some BMO analysis (cf. [8, Lemma 4 and Theorem 1]), one shows that there exists a constant $\varepsilon > 0$, which depends only on the bounds of the coefficients, the dimension, and $T$, such that

$$\mathbb{E}\Big\{\exp\Big(\varepsilon\int_0^T |\hat{Z}_t|^2 dt\Big) + |M_T^{\lambda\hat{Z}}|^{1+\varepsilon}\Big\} < \infty. \tag{4.7}$$

Consequently, $M^{\lambda\hat{Z}}$ is a true martingale.

Bearing this observation in mind we now give the main result of this section.

**Theorem 4.2** *Assume Assumption 4.1 holds.*

*(i) If the BSDE (3.6) has a solution $(\hat{Y}, \hat{Z})$ such that (3.8) holds, then the FBSDE (4.1) has a solution $\Theta \in \widehat{\mathbb{L}}_1$ such that $X \neq 0$ and (4.3) holds.*

*(ii) Conversely, if the FBSDE (4.1) has a solution $\Theta \in \widehat{\mathbb{L}}_1$ such that*

$$|Y_t| \le C|X_t|, \quad |X_t| \le C|X_t - \sigma_3 Y_t|, \tag{4.8}$$

*then $X \neq 0$, and the processes $(\hat{Y}, \hat{Z})$ defined by (4.2) satisfies BSDE (3.6) and (3.8).*

*Proof.* (i) In light of (4.3), we consider the following SDE:

$$\begin{aligned} dX_t &= X_t\Big[b_1 + b_2\hat{Y}_t + b_3\frac{\hat{Z}_t + \hat{Y}_t(\sigma_1 + \sigma_2\hat{Y}_t)}{1 - \sigma_3\hat{Y}_t}\Big]dt + X_t\Big[\sigma_1 + \sigma_2\hat{Y}_t + \sigma_3\frac{\hat{Z}_t + \hat{Y}_t(\sigma_1 + \sigma_2\hat{Y}_t)}{1 - \sigma_3\hat{Y}_t}\Big]dB_t \\ &= X_t\Big\{H_t(\hat{Y}_t, \hat{Z}_t)dt + [I_t(\hat{Y}_t) + \Lambda_t(\hat{Y}_t)\hat{Z}_t]dB_t\Big\}, \end{aligned} \tag{4.9}$$

where

$$H_t(y,z) \triangleq \Big[b_1 + b_2 y + b_3\frac{z + y(\sigma_1 + \sigma_2 y)}{1 - \sigma_3 y}\Big]; \qquad I_t(y) \triangleq \frac{\sigma_1 + \sigma_2 y}{1 - \sigma_3 y}.$$

It is then easy to check that

$$
\begin{aligned}
X_t &= \exp\Big\{\int_0^t [I_s(\hat Y_s)+\Lambda_s(\hat Y_s)\hat Z_s]dB_s + \int_0^t \Big[H_s(\hat Y_s,\hat Z_s) - \frac{1}{2}\big[I_s(\hat Y_s)+\Lambda_s(\hat Y_s)\hat Z_s\big]^2\Big]ds\Big\}\\
&= M_t^{\Lambda(\hat Y)\hat Z} M_t^{I(\hat Y)} \exp\Big\{\int_0^t [H_s(\hat Y_s,\hat Z_s) - I_s(\hat Y_s)\Lambda_s(\hat Y_s)\hat Z_s)]ds\Big\}. \qquad (4.10)
\end{aligned}
$$

Clearly $X > 0$. Furthermore, since (3.8) implies that in (4.9) $\Lambda(\hat Y)$, $I(\hat Y)$ are bounded and $H(\hat Y,\hat Z)$ has a linear growth in $\hat Z$, and (4.7) implies

$$
\mathbb{E}\Big\{\sup_{0\le t\le T}|M_t^{I(\hat Y)}|^p + \exp\Big(p\int_0^T [1+|\hat Z_t|]dt\Big)\Big\} < \infty \quad \text{for any} \quad p>1, \qquad (4.11)
$$

we deduce from (4.10) that, for $\varepsilon$ in (4.7) (noting that $(\frac{2(1+\varepsilon)}{2+\varepsilon}, \frac{2(1+\varepsilon)}{\varepsilon})$ are conjugates),

$$
\begin{aligned}
\mathbb{E}\Big\{\sup_{0\le t\le T}|X_t|^{1+\frac{\varepsilon}{2}}\Big\} &\le \Big(E\Big\{\sup_{0\le t\le T}|M_t^{\Lambda(\hat Y)\hat Z}|^{1+\varepsilon}\Big\}\Big)^{\frac{2+\varepsilon}{2(1+\varepsilon)}} \times \qquad (4.12)\\
&\qquad \Big(E\Big\{\sup_{0\le t\le T}\Big|M_t^{I(\hat Y)}\Big|^{\frac{(2+\varepsilon)(1+\varepsilon)}{\varepsilon}} e^{\frac{C(2+\varepsilon)(1+\varepsilon)}{\varepsilon}\int_0^T[1+|\hat Z_t|]dt}\Big\}\Big)^{\frac{\varepsilon}{2(1+\varepsilon)}} < \infty.
\end{aligned}
$$

Now if we define $(Y,Z)$ by (4.3), then $\Theta$ satisfy (4.1) and, by (4.7) again,

$$
\mathbb{E}\Big\{\sup_{0\le t\le T}|Y_t|^{1+\frac{\varepsilon}{2}} + \Big(\int_0^T |Z_t|^2dt\Big)^{1+\frac{\varepsilon}{4}}\Big\} < \infty. \qquad (4.13)
$$

That is, $\Theta \in \mathbb{L}^{1+\frac{\varepsilon}{4}} \subset \widehat{\mathbb{L}}_1$, proving (i).

(ii) We now assume that FBSDE (4.1) has a solution $\Theta \in \widehat{\mathbb{L}}_1$ such that (4.8) holds. Denote $\tau_n \stackrel{\triangle}{=} \inf\{t : X_t = \frac{1}{n}\}\wedge T$, $\tau \stackrel{\triangle}{=} \inf\{t : X_t = 0\}\wedge T$, and define $\hat Y, \hat Z$ by (4.2). Clearly, the assumption (4.8) implies that $\hat Y$ satisfies (3.8) in $[0,\tau)$, and applying Itô's formula we see that $(\hat Y,\hat Z)$ satisfies

$$
d\hat Y_t = -\Big[F_t(\hat Y_t) + G_t(\hat Y_t)\hat Z_t + \Lambda_t(\hat Y_t)|\hat Z_t|^2\Big]dt + \hat Z_t dB_t, \quad t\in[0,\tau).
$$

Note that the boundedness of $\hat Y$ implies that the above SDE is actually of the form of (4.6), and at least on $[0,\tau_n)$ the stochastic integral $\int_0^\cdot \hat Z_s dB_s$ is a true martingale. Thus we can apply the same argument there to obtain the bound (4.7) on $[0,\tau_n)$:

$$
\mathbb{E}\Big\{\exp\Big(\varepsilon\int_0^{\tau_n}|\hat Z_t|^2dt\Big) + |M_{\tau_n}^{\Lambda(\hat Y)\hat Z}|^{1+\varepsilon}\Big\} \le C < \infty.
$$

Note that the constants $\varepsilon$ and $C$ above depends on the coefficients, which depends only on the bound of $\hat Y$ and is independent of $n$, thanks to (4.8). Thus, letting $n\to\infty$ we have

$$
\mathbb{E}\Big\{\exp\Big(\varepsilon\int_0^{\tau}|\hat Z_t|^2dt\Big) + |M_{\tau}^{\Lambda(\hat Y)\hat Z}|^{1+\varepsilon}\Big\} \le C < \infty.
$$

On the other hand, since $X$ satisfies (4.10) on $[0,\tau)$, we see that the estimate above implies that $X_\tau > 0$, a.s. Thus $\tau = T$ a.s. In other words, $(\hat{Y}, \hat{Z})$ satisfies (3.6) over $[0,T]$, and (3.8) holds. The proof is now complete. ■

We conclude this section by presenting a result regarding the uniqueness of the solutions to FBSDE (4.1) and its characteristic BSDE (3.6). Such a result, although will not be used in the rest of the paper, is interesting in its own right. To this end we need an additional condition on $(\hat{Y}, \hat{Z})$ that strengthen the estimates (4.7):

$$\mathbb{E}\Big\{\sup_{0\le t\le T} |M_t^{\Lambda(\hat{Y})\hat{Z}}|^{2+\varepsilon}\Big\} < \infty \quad \text{for some} \quad \varepsilon > 0. \tag{4.14}$$

**Theorem 4.3** *Assume that Assumption 4.1 holds. Then the BSDE (3.6) has a solution $(\hat{Y}, \hat{Z})$ satisfying (3.8) and (4.14) if and only if the FBSDE (4.1) has a solution $\Theta \in \widehat{\mathbb{L}}_2$ satisfying (4.8).*

*Moreover, in such a case the uniqueness holds for solutions to BSDE (3.6) satisfying (3.8) and (4.14) with $\varepsilon = 0$ and for solutions to FBSDE (4.1) in $\mathbb{L}^2$ satisfying (4.8).*

*Proof.* We first assume that that (3.6) has a solution $(\hat{Y}, \hat{Z})$ that satisfies (3.8) and (4.14). Then by Theorem 4.2, the FBSDE (4.1) has a solution $\Theta \in \widehat{\mathbb{L}}_1$. Furthermore, using condition (4.14) we can actually improve the estimates (4.12) and (4.13) to $\mathbb{L}^{2+\varepsilon/2}$, and consequently $\Theta \in \widehat{\mathbb{L}}_2$.

Conversely, if FBSDE (4.1) has a solution $\Theta \in \widehat{\mathbb{L}}_2 \subseteq \widehat{\mathbb{L}}_1$, then by Theorem 4.2 (ii), the $(\hat{Y}, \hat{Z})$ defined by (4.2) satisfy (3.6) and (3.8), and $X$ satisfy (4.10). Thus

$$M_t^{\Lambda(\hat{Y})\hat{Z}} \quad = \quad X_t[M_t^{I(\hat{Y})}]^{-1} \exp\Big\{-\int_0^t [H_s(\hat{Y}_s, \hat{Z}_s) - I_s(\hat{Y}_s)\Lambda_s(\hat{Y}_s)\hat{Z}_s)]ds\Big\}.$$

If $\mathbb{E}\{\sup_{0\le t\le T} |X_t|^p\} < \infty$ for some $p > 2$, by estimates similar to (4.12) we obtain (4.14).

We now turn our attention to the uniqueness. Let $\Theta \in \mathbb{L}^2$ be an arbitrary solution to FBSDE (4.1) satisfying (4.8) and $(\hat{Y}, \hat{Z})$ be an arbitrary solution to BSDE (3.6) satisfying (3.8) and (4.14) with $\varepsilon = 0$. We claim that, if $\Theta \in \widehat{\mathbb{L}}_2$ or $(\hat{Y}, \hat{Z})$ satisfies (4.14) with some $\varepsilon > 0$, then relation (4.2) and equivalently (4.3) must hold. Then on one hand, fix one solution $\Theta \in \widehat{\mathbb{L}}_2$ of FBSDE (1.1) satisfying (4.8). For any solution $(\hat{Y}, \hat{Z})$ to BSDE (3.6) satisfying (3.8) and (4.14) with $\varepsilon = 0$, by (4.2) we see that $(\hat{Y}, \hat{Z})$ is unique. On the other hand, for the given (unique) $(\hat{Y}, \hat{Z})$ satisfying (3.8) and (4.14) with $\varepsilon > 0$, for any solution $\Theta \in \mathbb{L}^2$ of FBSDE (4.1) satisfying (4.8) , by (4.3) we see that $X$ must satisfy (4.10) and thus is unique. Then the uniqueness of $(Y, Z)$ follows from (4.3).

We now prove (4.2). Given $\Theta$ and $(\hat{Y}, \hat{Z})$, denote

$$\Delta Y_t \stackrel{\triangle}{=} Y_t - \hat{Y}_t X_t, \quad \Delta Z_t \stackrel{\triangle}{=} Z_t - \Big[X_t \hat{Z}_t + \hat{Y}_t(\sigma_1 X_t + \sigma_2 Y_t + \sigma_3 Z_t)\Big]. \tag{4.15}$$

Applying Ito's formula to $\Delta Y_t$ we have

$$\begin{aligned} d(\Delta Y_t) &= -\Big[f_1 X_t + f_2 Y_t + f_3 Z_t - X_t[F_t(\hat{Y}_t) + G_t(\hat{Y}_t)\hat{Z}_t + \Lambda_t(\hat{Y}_t)|\hat{Z}_t|^2] \\ &\quad + \hat{Y}_t(b_1 X_t + b_2 Y_t + b_3 Z_t) + \hat{Z}_t(\sigma_1 X_t + \sigma_2 Y_t + \sigma_3 Z_t)\Big]dt + \Delta Z_t dB_t \\ &= -\Big[X_t\big[f_1 - F_t(\hat{Y}_t) - G_t(\hat{Y}_t)\hat{Z}_t - \Lambda_t(\hat{Y}_t)|\hat{Z}_t|^2 + b_1\hat{Y}_t + \sigma_1\hat{Z}_t\big] \qquad (4.16) \\ &\quad + Y_t\big[f_2 + b_2\hat{Y}_t + \sigma_2\hat{Z}_t\big] + Z_t\big[f_3 + b_3\hat{Y}_t + \sigma_3\hat{Z}_t\big]\Big]dt + \Delta Z_t dB_t. \end{aligned}$$

By (4.15), one can easily check that

$$Y = \Delta Y + \hat{Y}X, \quad Z = \frac{\Delta Z + X\hat{Z} + \hat{Y}(\sigma_1 X + \sigma_2 Y)}{1 - \sigma_3\hat{Y}} = \frac{\Delta Z + \sigma_2\hat{Y}\Delta Y + X[\hat{Z} + (\sigma_1 + \sigma_2\hat{Y})\hat{Y}]}{1 - \sigma_3\hat{Y}}.$$

Plugging these into (4.16) we obtain

$$d(\Delta Y_t) = -\Big[\alpha_t X_t + \beta_t \Delta Y_t + \gamma_t \Delta Z_t\Big]dt + \Delta Z_t dB_t,$$

where

$$\begin{aligned} \gamma_t &\stackrel{\triangle}{=} \frac{f_3 + b_3\hat{Y}_t + \sigma_3\hat{Z}_t}{1 - \sigma_3\hat{Y}_t} = \frac{f_3 + b_3\hat{Y}_t}{1 - \sigma_3\hat{Y}_t} + \Lambda_t(\hat{Y}_t)\hat{Z}_t; \\ \beta_t &\stackrel{\triangle}{=} f_2 + b_2\hat{Y}_t + \sigma_2\hat{Z}_t + \frac{\sigma_2\hat{Y}_t[f_3 + b_3\hat{Y}_t + \sigma_3\hat{Z}_t]}{1 - \sigma_3\hat{Y}_t}; \end{aligned} \qquad (4.17)$$

and

$$\begin{aligned} \alpha_t &\stackrel{\triangle}{=} f_1 - F_t(\hat{Y}_t) - G_t(\hat{Y}_t)\hat{Z}_t - \Lambda_t(\hat{Y}_t)|\hat{Z}_t|^2 + b_1\hat{Y}_t + \sigma_1\hat{Z}_t \\ &\quad + [f_2 + b_2\hat{Y}_t + \sigma_2\hat{Z}_t]\hat{Y}_t + [f_3 + b_3\hat{Y}_t + \sigma_3\hat{Z}_t]\frac{\hat{Z}_t + (\sigma_1 + \sigma_2\hat{Y}_t)\hat{Y}_t}{1 - \sigma_3\hat{Y}_t} = 0, \end{aligned}$$

thanks to (3.7). Denote

$$\Gamma_t \stackrel{\triangle}{=} M_t^{\gamma}\exp\Big(\int_0^t \beta_s ds\Big) = M_t^{\Lambda(\hat{Y})\hat{Z}} M_t^{\frac{f_3 + b_3\hat{Y}}{1 - \sigma_3\hat{Y}}}\exp\Big(\int_0^t [\beta_s - \frac{f_3 + b_3\hat{Y}_s}{1 - \sigma_3\hat{Y}_s}\Lambda_s(\hat{Y}_s)\hat{Z}_s]ds\Big). \qquad (4.18)$$

Then by applying Itô's formula one obtains immediately

$$d(\Gamma_t \Delta Y_t) = \Gamma_t[\gamma_t \Delta Y_t + \Delta Z_t]dB_t. \qquad (4.19)$$

We claim that

$$\mathbb{E}\Big\{\Big(\int_0^T |\Gamma_t|^2[\gamma_t\Delta Y_t + \Delta Z_t]^2 dt\Big)^{\frac{1}{2}}\Big\} \le \mathbb{E}\Big\{\sup_{0\le t\le T}|\Gamma_t|\Big(\int_0^T [\gamma_t\Delta Y_t + \Delta Z_t]^2 dt\Big)^{\frac{1}{2}}\Big\} < \infty, \qquad (4.20)$$

so that $\int_0^\cdot \Gamma_s[\gamma_t\Delta Y_t + \Delta Z_s]dB_s$ is a true martingale. Since $\Delta Y_T = 0$ and $\Gamma_0 = 1$, it follows from (4.19) that $\Delta Y = 0$, and hence $\Delta Z = 0$. Then (4.15) leads to (4.2) immediately.

It remains to prove (4.20). Note that

$$|\gamma_t| \le C[1+|\hat{Z}_t|], \quad |\Delta Y_t| \le C[|X_t|+|Y_t|], \quad |\Delta Z_t| \le C[|X_t|+|Y_t|+|Z_t|+|X_t||\hat{Z}_t|].$$

Then

$$\begin{aligned} &\int_0^T [\gamma_t \Delta Y_t + \Delta Z_t]^2 dt \\ \le\ & C\Big[1+\sup_{0\le t\le T}[|X_t|^2+|Y_t|^2] + \int_0^T [|Z_t|^2+|\hat{Z}_t|^2]dt + \sup_{0\le t\le T}|X_t|^2 \int_0^T |\hat{Z}_t|^2 dt\Big]. \end{aligned} \tag{4.21}$$

Since $(\hat{Y},\hat{Z})$ satisfies (3.8), by (4.7) we have

$$\mathbb{E}\Big\{\Big(\int_0^T |\hat{Z}_t|^2 dt\Big)^p\Big\} < \infty \quad \text{for any} \quad p\ge 1. \tag{4.22}$$

In the case that (4.14) holds with some $\varepsilon > 0$, following the arguments for (4.12) we have

$$\mathbb{E}\{\sup_{t\in[0,T]} |\Gamma_t|^{2+\frac{\varepsilon}{2}}\} < \infty. \tag{4.23}$$

Then for any $\Theta \in \mathbb{L}^2$, plugging (4.22) and (4.23) into (4.21) we have (4.20) immediately.

In the case that $\Theta \in \widehat{\mathbb{L}}_2$, we may assume $\Theta \in \mathbb{L}^{2+\varepsilon}$. Note that

$$\frac{1}{2+\varepsilon} + \frac{3+2\varepsilon}{6+3\varepsilon} + \frac{\varepsilon}{6+3\varepsilon} = 1 \quad \text{and} \quad \frac{6+3\varepsilon}{3+2\varepsilon} < 2.$$

Since (4.14) holds with $\varepsilon = 0$, following the arguments for (4.12) we have

$$\mathbb{E}\{\sup_{t\in[0,T]} |\Gamma_t|^{\frac{6+3\varepsilon}{3+2\varepsilon}}\} < \infty.$$

This implies that

$$\begin{aligned} &\mathbb{E}\Big\{\sup_{0\le t\le T}|\Gamma_t| \sup_{0\le t\le T}|X_t| \int_0^T |\hat{Z}_t|^2 dt\Big\} \\ \le\ & \Big(\mathbb{E}\Big\{\sup_{0\le t\le T}|\Gamma_t|^{\frac{6+3\varepsilon}{3+2\varepsilon}}\Big\}\Big)^{\frac{3+2\varepsilon}{6+3\varepsilon}} \Big(\mathbb{E}\Big\{\sup_{0\le t\le T}|X_t|^{2+\varepsilon}\Big\}\Big)^{\frac{1}{2+\varepsilon}} \Big(\mathbb{E}\Big\{\Big(\int_0^T |\hat{Z}_t|^2 dt\Big)^{\frac{6+3\varepsilon}{\varepsilon}}\Big\}\Big)^{\frac{\varepsilon}{6+3\varepsilon}} < \infty. \end{aligned}$$

Then one can easily prove (4.20) again. ■

# 5 Wellposedness of the Dominating Equation

We note that Theorems 4.2 and 4.3 only established the relations of the wellposedness between the characteristic BSDEs and the original FBSDE, it does not provide the wellposedness result

for either one of them. In this section we take a closer look at the dominating ODEs (3.12). Since the existence of desirable bounded solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ to the dominating ODEs will lead to the wellposedness of the characteristic BSDEs, which will eventually lead to that of the original FBSDE (1.1), the results in this section will help us to establish a *user's guide* in the end.

We begin with a special form of comparison theorem among the solutions to ODEs. Consider the following "backward ODEs" on $[0,T]$:

$$\mathbf{y}_t^0 = h^0 + \int_t^T F^0(s, \mathbf{y}_s^0) ds; \tag{5.1}$$

and

$$\mathbf{y}_t^1 = h^1 - C^1 + \int_t^T [F^1(s, \mathbf{y}_s^1) + c_s^1] ds, \quad \mathbf{y}_t^2 = h^2 + C^2 + \int_t^T [F^2(s, \mathbf{y}_s^2) - c_s^2] ds. \tag{5.2}$$

where $F^0, F^1, F^2 : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$ are (deterministic) measurable function. The following simple lemma will be useful in our discussion. Its proof is rather elementary and we defer it to the Appendix.

**Lemma 5.1** *Assume that*

*(i) $h^1 \le h^0 \le h^2$, and $F_1 \le F^0 \le F_2$;*

*(ii) Both ODEs in* (5.2) *admit bounded solutions $\mathbf{y}^1$ and $\mathbf{y}^2$ on $[0,T]$.*

*(iii) For each $t \in [0,T]$, the function $y \mapsto F(t,y)$ is uniformly Lipschitz continuous for $y \in [\mathbf{y}_t^1, \mathbf{y}_t^2]$, with a common Lipschitz constant $L$.*

*(iv) $C^i \ge \int_t^T e^{-\int_s^T \alpha_r dr} c_s^i ds$, for all $t \in [0,T]$ and all $\alpha$ satisfying $|\alpha| \le L$.*

*Then (5.1) has a unique solution $\mathbf{y}^0$ satisfying $\mathbf{y}^1 \le \mathbf{y}^0 \le \mathbf{y}^2$.*

**Remark 5.2** Two typical sufficient conditions for condition (iv) in the above are:

(i) $C^i = 0$ and $c^i \le 0$;

(ii) $C^i \ge \int_0^T e^{L(T-t)} (c_t^i)^+ dt$. ■

## 5.1 Linear FBSDE with constant coefficients

We first investigate the linear FBSDE (4.1) where all the coefficients are constants. We shall show that in such a case some "sharp" (sufficient and necessary) conditions regarding well-posedness can be obtained. These results, to our best knowledge, are novel in the literature; and at the same time, they more or less set the "limits" for the solvability of general FBSDE (1.1).

We carry out our analysis in two cases.

**Case 1.** $\sigma_3 = 0$.

In this case $\overline{h} = \underline{h} = h$, $\overline{F}(t,y) = \underline{F}(t,y) = F(y)$, and the two ODEs in (3.12) become the same:

$$\mathbf{y}_t = h + \int_t^T F(\mathbf{y}_s)ds \tag{5.3}$$

where

$$F(y) = f_1 + [\sigma_1 f_3 + b_1 + f_2]y + [b_3\sigma_1 + \sigma_2 f_3 + b_2]y^2 + \sigma_2 b_3 y^3. \tag{5.4}$$

We have the following theorem.

**Theorem 5.3** *Assume that in the linear FBSDE (4.1) all coefficients are constants, and $\sigma_3 = 0$. Then the corresponding dominating ODE (5.3) with $F$ defined by (5.4) has a bounded solution for arbitrary $T$ if and only if one of the following three cases hold true:*

*(i) $F(h) \geq 0$ and $F$ has a zero point in $[h, \infty)$;*

*(ii) $F(h) \leq 0$ and $F$ has a zero point in $(-\infty, h]$,*

*(iii) $b_3\sigma_2 = 0$ and $b_2 + f_3\sigma_2 + b_3\sigma_1 = 0$.*

*Proof.* We first prove the sufficiency part. In case (i), there exists $\lambda \geq h$ such that $F(\lambda) = 0$. Note that $F$ is locally Lipschitz continuous in $y$ and

$$h = h + \int_t^T [F(h) - F(h)]ds, \quad \lambda = \lambda + \int_t^T F(\lambda)ds.$$

Then it follows from Lemma 5.1 and in particular Remark 5.2 (i) that $\mathbf{y}_t \in [h, \lambda]$, $t \in [0,T]$. Similarly in case (ii) one has $\mathbf{y}_t \in [\lambda, h]$, for some $\lambda \leq h$ such that $F(\lambda) = 0$. Finally, in case (iii) the ODE (5.3) becomes linear:

$$\mathbf{y}_t = h + \int_t^T [f_1 + (f_2 + b_1 + f_3\sigma_1)\mathbf{y}_s]ds. \tag{5.5}$$

Thus it is obviously bounded.

The proof of necessity is elementary but lengthy. Since it will not be used for the rest of the paper, we postpone its proof to Appendix. ■

When the terminal time $T > 0$ is fixed, we have the following slightly weaker sufficient conditions:

**Theorem 5.4** *For any given $T > 0$, the ODE (5.3) with $F$ given in (5.4) has a bounded solution on $[0,T]$ if one of the following three cases hold true:*

*(i) $\sigma_2 b_3 < 0$ or $F(h) = 0$;*

*(ii) $F(h) > 0$, and there exists a constant $\varepsilon = \varepsilon(T) > 0$ small enough, such that*

$$\sigma_2 b_3 \leq \varepsilon \quad \textit{and} \quad b_2 + f_3\sigma_2 + b_3\sigma_1 \leq \varepsilon.$$

*(ii) $F(h) < 0$, and there exists a constant $\varepsilon = \varepsilon(T) > 0$ small enough such that*

$$\sigma_2 b_3 \leq \varepsilon \quad \textit{and} \quad b_2 + f_3\sigma_2 + b_3\sigma_1 \geq -\varepsilon.$$

*Proof.* (i) In this case clearly the result follows from either (i) or (ii) of Theorem 5.3.

(ii) In this case we have, for some constants $C_1, C_0$,

$$F(y) \leq \varepsilon y^3 + \varepsilon y^2 + C_1 y + C \leq 2\varepsilon y^3 + C_1 y + C_0, \quad \text{for all} \quad y \geq 0. \tag{5.6}$$

We first solve

$$\tilde{\mathbf{y}}_t = h^+ + \int_t^T [C_1\tilde{\mathbf{y}}_s + C_0 + 1]ds,$$

and obtain

$$\tilde{\mathbf{y}}_t = e^{C_1(T-t)}h^+ + \frac{C_0+1}{C_1}[e^{C_1(T-t)} - 1] \leq C_2 := e^{C_1 T}h^+ + \frac{C_0+1}{C_1}[e^{C_1 T} - 1]. \tag{5.7}$$

Set $\varepsilon \stackrel{\triangle}{=} \frac{1}{2C_2^3}$ so that $2\varepsilon\tilde{\mathbf{y}}_t^3 \leq 1$. Note that

$$\tilde{\mathbf{y}}_t = h^+ + \int_t^T [2\varepsilon\tilde{\mathbf{y}}_s^3 + C_1\tilde{\mathbf{y}}_s + C_0 + (1 - 2\varepsilon\tilde{\mathbf{y}}_s^3)]ds.$$

By (5.6), applying Lemma 5.1 and in particular Remark 5.2 (i) we see that ODE (5.3) has a solution $\mathbf{y} \in [h, \tilde{\mathbf{y}}] \subset [h, C_2]$.

(iii) can be proved similarly. ■

**Case 2.** $\sigma_3 \neq 0$.

In this case we still have $\overline{h} = \underline{h} = h$, $\overline{F}(t,y) = \underline{F}(t,y) = F(y)$, where the deterministic function $F$ in (5.4) can be rewritten as

$$F(y) = \frac{\alpha_0}{\frac{1}{\sigma_3} - y} + \alpha_1 + \alpha_2 y + [b_2 - \frac{b_3\sigma_2}{\sigma_3}]y^2, \tag{5.8}$$

for some constants $\alpha_0, \alpha_1, \alpha_2$. In this case the two ODEs in (3.12) also become the same one (5.3) and, in light of (3.8), we want to find its solution satisfying that

$$\text{both } \mathbf{y} \text{ and } (1 - \sigma_3\mathbf{y})^{-1} \text{ are bounded.} \tag{5.9}$$

In particular, this requires that $\sigma_3 h \neq 1$. We remark that, when $\sigma_3 h = 1$, there are counter examples in both existence and uniqueness of the linear FBSDE (4.1) (cf., e.g., [15]). We now have

**Theorem 5.5** *Assume the FBSDE is the linear one (4.1) and all the coefficients are constants. Assume also that $\sigma_3 \neq 0$ and $h\sigma_3 \neq 1$. Then the ODE (5.3) has a solution satisfying (5.9) for arbitrary $T$ if and only if one of the following four cases holds:*

*(i) $h < \frac{1}{\sigma_3}$, $F(h) \leq 0$, and either $F$ has a zero point in $(-\infty, h]$ or $b_2 - \frac{b_3\sigma_2}{\sigma_3} = 0$;*

*(ii) $h > \frac{1}{\sigma_3}$, $F(h) \geq 0$, and either $F$ has a zero point in $[h, \infty)$ or $b_2 - \frac{b_3\sigma_2}{\sigma_3} = 0$;*

*(iii) $h < \frac{1}{\sigma_3}$, $F(h) \geq 0$, and $F$ has a zero point in $[h, \frac{1}{\sigma_3})$;*

*(iv) $h > \frac{1}{\sigma_3}$, $F(h) \leq 0$, and $F$ has a zero point in $(\frac{1}{\sigma_3}, h]$.*

*Proof.* We first prove the sufficiency.

(i) If $F(\lambda) = 0$ for some $\lambda \in (-\infty, h]$, then as in Theorem 5.3 we see that ODE (5.3) has a solution $\mathbf{y} \in [\lambda, h]$. Thus (5.9) holds. We now assume instead that $b_2 - \frac{b_3\sigma_2}{\sigma_3} = 0$. Then from (5.8) we see that $F(y) = \alpha_0(\frac{1}{\sigma_3} - y)^{-1} + \alpha_1 + \alpha_2 y$. Consider

$$\tilde{\mathbf{y}}_t = h + \int_t^T [-|\alpha_0|(\frac{1}{\sigma_3} - h)^{-1} + \alpha_1 + \alpha_2 \tilde{\mathbf{y}}_s] ds.$$

Since $F(h) \leq 0$, clearly the above SDE has a bounded solution $\tilde{\mathbf{y}} \leq h$. Applying Lemma 5.1 one can easily see that (5.3) has a solution $\mathbf{y} \in [\tilde{\mathbf{y}}, h]$. Thus (5.9) holds.

(ii) can be proved similarly, and (iii) and (iv) are obvious now.

The necessary part is again postponed to Appendix. ∎

When $T$ is fixed, we may also have some slightly weaker sufficient conditions. However, these conditions are more involved, so we omit them here and will discuss directly for the general case in next subsection, see Theorems 5.7 and 5.8 below.

## 5.2 The nonlinear case

Again we consider the case that $\sigma_3 = 0$ first.

**Case 1.** $\sigma = \sigma(t, x, y)$.

We recall that in this case $F$ takes the form (5.4), where $b_i$, $\sigma_i$, $f_i$, $i = 1, 2, 3$, are bounded, adapted processes defined by (3.3), and thus $F$ is also random and may depend on $t$. Now recall the definition of the functions $\overline{F}$ and $\underline{F}$ (3.11). Again, by a slight abuse of notations we replace $\Theta^j$, $j = 1, 2$ in (3.3) by $\theta_j$, $j = 1, 2$, and still denote them by $b_i$, $\sigma_i$, $f_i$, $i = 1, 2, 3$. In what follows all assumptions involving coefficients in (5.4) will be in the sense that they hold uniformly for all $\theta_j$, $j = 1, 2$. In analogy to Theorem 5.4, we have the following result.

**Theorem 5.6** *Assume Assumption 2.1 holds and $\sigma = \sigma(t, x, y)$. Then, for any $T > 0$, the ODEs (3.12) have bounded solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ on $[0, T]$ if one of the following three cases holds true:*

*(i) there exists a constant $\varepsilon > 0$ such that,*

$$\sigma_2 b_3 \le -\varepsilon. \tag{5.10}$$

*(ii) there exists a constant $\lambda \le \underline{h}$, and a constant $\varepsilon > 0$ small enough such that*

$$\underline{F}(t,\lambda) \ge 0, \quad \sigma_2 b_3 \le \varepsilon \quad \text{and} \quad b_2 + f_3\sigma_2 + b_3\sigma_1 \le \varepsilon. \tag{5.11}$$

*(iii) there exists a constant $\lambda \ge \overline{h}$, and a constant $\varepsilon > 0$ small enough such that*

$$\overline{F}(t,\lambda) \le 0, \quad \sigma_2 b_3 \le \varepsilon \quad \text{and} \quad b_2 + f_3\sigma_2 + b_3\sigma_1 \ge -\varepsilon. \tag{5.12}$$

*Proof.* (i) In this case we have

$$\overline{F}(t,y) \le -\varepsilon y^3 + C[y^2+1] \ \text{ for all } \ y \ge 0, \quad \text{and} \quad \underline{F}(t,y) \ge -\varepsilon y^3 - C[y^2+1] \ \text{ for all } \ y \le 0.$$

Then there exist $\lambda_1 \in (-\infty, \underline{h} \wedge 0]$ and $\lambda_2 \in [\overline{h} \vee 0, +\infty)$ such that $\underline{F}(t,\lambda_1) \ge 0$, $\overline{F}(t,\lambda_2) \le 0$, uniformly for all $t \in [0,T]$. Since $\underline{h} \le \overline{h}$ and $\underline{F} \le \overline{F}$, we must have

$$\underline{F}(t,\lambda_2) \le \overline{F}(t,\lambda_2) \le 0 \le \underline{F}(t,\lambda_1), \quad \overline{F}(t,\lambda_2) \le 0 \le \underline{F}(t,\lambda_1) \le \overline{F}(t,\lambda_1).$$

Now applying Lemma 5.1, we conclude that the ODEs in (3.12) have bounded solutions $\overline{\mathbf{y}}_t \in [\lambda_1, \lambda_2]$ and $\underline{\mathbf{y}}_t \in [\lambda_1, \lambda_2]$, respectively.

(ii) In this case, similar to (5.6) we have

$$\overline{F}(t,\lambda) \ge \underline{F}(t,\lambda) \ge 0, \quad \text{and} \quad \underline{F}(t,y) \le \overline{F}(t,y) \le 2\varepsilon y^3 + C_1 y + C_0, \quad \text{for all } \ y \ge 0.$$

Let $C_2$ be defined by (5.7) and set $\varepsilon := \frac{1}{2C_2^3}$. Follow the arguments in Theorem 5.4 (ii), we see that the ODEs in (3.12) have bounded solutions $\lambda \le \underline{\mathbf{y}} \le \overline{\mathbf{y}} \le C_2$.

(iii) can be proved similarly. ∎

**Case 2.** $\sigma = \sigma(t,x,y,z)$**.**

This case has been avoided in many of the existing literature, especially when one uses the decoupling strategy. A well-known sufficient condition for the existence is, roughly speaking, that $|\sigma_3 h| < 1$. As we will see below, the condition we need is essentially $\sigma_3 h \ne 1$. In particular, we shall discuss three different cases:

(**2-a**) $|\sigma_3 h| < 1$;

(**2-b**) $\sigma_3 h > 1$ and both $\sigma_3$ and $h$ do not change sign;

**(2-c)** $\sigma_3 h < 1$ and either $\sigma_3$ or $h$ does not change sign.

We remark that, in the constant case, the above three cases (actually the latter two) cover all possible cases of $\sigma_3 h \neq 1$. However, for general nonlinear FBSDEs with random coefficients, we need them to hold uniformly in certain sense.

To be more precise, let $T > 0$ be given. We begin by assuming that there are three constants $c_1, c_2, c_3$ satisfying

$$c_1 > 0, \quad 0 < c_2 < c_3, \quad c_1 c_3 < 1. \tag{5.13}$$

The following result gives the answer to Case (2-a).

**Theorem 5.7** *Assume that Assumption 2.1 and (5.13) are in force. Assume also that there exists a constant $\varepsilon = \varepsilon(T) > 0$ small enough such that*

$$|\sigma_3| \le c_1, \quad |h| \le c_2, \quad \textit{and} \quad \overline{F}(t, c_3) \le \varepsilon, \quad \underline{F}(t, -c_3) \ge -\varepsilon, \tag{5.14}$$

*Then the ODEs in (3.12) have solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ satisfying*

$$-c_3 \le \underline{\mathbf{y}} \le \overline{\mathbf{y}} \le c_3 \quad \textit{and hence} \quad \textit{both } \overline{\mathbf{y}} \textit{ and } \underline{\mathbf{y}} \textit{ satisfy (5.9).}$$

*Proof.* Note that $1 - \sigma_3 y \ge 1 - c_1 c_3 > 0$ for $y \in [-c_3, c_3]$, then $\overline{F}$ and $\underline{F}$ are uniformly Lipschitz continuous in $y$ for $y \in [-c_3, c_3]$, and we denote by $L$ their uniform Lipschitz constant. Clearly $\tilde{\mathbf{y}}^1_t \stackrel{\triangle}{=} -c_3$ and $\tilde{y}^2_t \stackrel{\triangle}{=} c_3$ satisfy the following ODEs:

$$\begin{aligned}
\tilde{\mathbf{y}}^1_t &= -c_2 - (c_3 - c_2) + \int_t^T [\underline{F}(s, \tilde{\mathbf{y}}^1_s) - \underline{F}(s, -c_3)] ds, \\
\tilde{\mathbf{y}}^2_t &= c_2 + (c_3 - c_2) + \int_t^T [\overline{F}(s, \tilde{\mathbf{y}}^2_s) - \overline{F}(s, c_3)] ds.
\end{aligned}$$

Now set $\varepsilon > 0$ small enough such that

$$c_3 - c_2 > \int_0^T e^{L(T-t)} \varepsilon dt.$$

Then it follows from Lemma 5.1 and in particular Remark 5.2 (ii) we obtain the result. ■

We next consider Case (2-b).

**Theorem 5.8** *Let Assumption 2.1 and (5.13) hold. Assume that there exists a constant $\varepsilon > 0$ small enough such that one of the following four cases holds true:*

$$\sigma_3 \ge c_1^{-1}, \quad h \ge c_2^{-1}, \quad \textit{and} \quad \underline{F}(t, c_3^{-1}) \ge -\varepsilon, \quad b_2 - \frac{b_3 \sigma_2}{\sigma_3} \le \varepsilon; \tag{5.15}$$

$$\sigma_3 \le -c_1^{-1}, \quad h \ge c_2^{-1}, \quad \textit{and} \quad \underline{F}(t, c_3^{-1}) \ge -\varepsilon, \quad b_2 - \frac{b_3 \sigma_2}{\sigma_3} \le \varepsilon; \tag{5.16}$$

$$\sigma_3 \ge c_1^{-1}, \quad h \le -c_2^{-1}, \quad \textit{and} \quad \overline{F}(t, -c_3^{-1}) \le \varepsilon, \quad b_2 - \frac{b_3 \sigma_2}{\sigma_3} \ge -\varepsilon; \tag{5.17}$$

$$\sigma_3 \le -c_1^{-1}, \quad h \le -c_2^{-1}, \quad \textit{and} \quad \overline{F}(t, -c_3^{-1}) \le \varepsilon, \quad b_2 - \frac{b_3 \sigma_2}{\sigma_3} \ge -\varepsilon. \tag{5.18}$$

*Then the ODEs in (3.12) have bounded solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ such that they satisfy the corresponding property of $h$ in the above conditions with $c_2$ being replaced by $c_3$. In particular, both $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ satisfy (5.9).*

*Proof.* We prove only the case (5.15). The other cases can be proved similarly.

In this case we have

$$\overline{F}(t,y) \le \frac{C}{c_3^{-1} - c_1} + C_1 y + \varepsilon y^2 = C_0 + C_1 y + \varepsilon y^2, \quad \text{for all } y \ge c_3^{-1}.$$

Let $\tilde{\mathbf{y}}$ denote the bounded solution to the following ODE:

$$\tilde{\mathbf{y}}_t = \overline{h} + \int_t^T [C_1 \tilde{\mathbf{y}}_s + C_0 + 1] ds \quad \text{and} \quad C_2 := \tilde{\mathbf{y}}_0 = \sup_{0 \le t \le T} \tilde{\mathbf{y}}_t.$$

Let $L$ denote the uniform Lipschitz constant of $\underline{F}$ and $\overline{F}$ for $y \in [c_3^{-1}, C_2]$. Note that

$$\underline{F}(t, c_3^{-1}) \ge -\varepsilon.$$

Now follow the arguments in Theorem 5.7 for the lower bound and those in Theorem 5.4 (ii) for the upper bound, one can easily show that, for $\varepsilon$ sufficiently small, the ODEs in (3.12) have solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ such that $c_3^{-1} \le \underline{\mathbf{y}} \le \overline{\mathbf{y}} \le C_2$. ■

We finally present the result for Case (2-c).

**Theorem 5.9** *Let Assumption 2.1 and (5.13) hold. Assume there exists a constant $\varepsilon > 0$ small enough such that one of the following four cases holds true:*

$$\sigma_3 \le c_1, \quad 0 \le h \le c_2, \quad \textit{and} \quad \overline{F}(t, c_3) \le \varepsilon, \quad f_1 \ge 0; \tag{5.19}$$

$$0 \le \sigma_3 \le c_1, \quad h \le c_2, \quad \textit{and} \quad \overline{F}(t, c_3) \le \varepsilon, \quad b_2 - \frac{b_3 \sigma_2}{\sigma_3} \ge -\varepsilon; \tag{5.20}$$

$$\sigma_3 \ge -c_1, \quad 0 \ge h \ge -c_2, \quad \textit{and} \quad \underline{F}(t, -c_3) \ge -\varepsilon, \quad f_1 \le 0; \tag{5.21}$$

$$0 \ge \sigma_3 \ge -c_1, \quad h \ge -c_2 \quad \textit{and} \quad \underline{F}(t, -c_3) \ge -\varepsilon, \quad b_2 - \frac{b_3 \sigma_2}{\sigma_3} \le \varepsilon. \tag{5.22}$$

*Then the ODEs in (3.12) have bounded solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ such that they satisfy the corresponding property of $h$ in the above conditions with $c_2$ being replaced by $c_3$. In particular, both $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ satisfy (5.9).*

*Proof.* If (5.19) holds, then

$$\underline{F}(t, 0) \ge 0, \quad \overline{F}(t, c_3) \le \varepsilon.$$

Follow the arguments in Theorem 5.3 for the lower bound and those in Theorem 5.6 for the upper bound, one can easily show that, for $\varepsilon$ sufficiently small, the ODEs in (3.12) have solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ such that $0 \le \underline{\mathbf{y}} \le \overline{\mathbf{y}} \le c_3$.

If (5.20) holds, follow the arguments in Theorem 5.4 (ii) for the lower bound and those in Theorem 5.6 for the upper bound, one can easily show that, for $\varepsilon$ sufficiently small, the ODEs in (3.12) have solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ such that $-C_2 \leq \underline{\mathbf{y}} \leq \overline{\mathbf{y}} \leq c_3$ for some $C_2 > 0$.

The other two cases can be proved similarly. ■

## 6 Small Duration Case Revisited

At this point we would like to point out that the well-posedness of (linear) variational FBSDE (1.3) and the associated *characteristic BSDE* (3.6) studied in previous sections only guarantee the existence of the decoupling field, assuming that the existence of the solution to the original FBSDE (1.1). In fact, the coefficients of (1.3) actually depend on the solution to the FBSDE. Therefore, except for the case when the FBSDE (1.1) itself is linear, the starting point should be the "local existence" result for FBSDE, or more precisely, the wellposedness of FBSDE (1.1) over small time interval. We note that this seemingly well-understood problem still contains many interesting issues that have not been completely observed, especially in the case when $\sigma$ depends on $z$ (i.e., $\sigma_3 \neq 0$), which we now describe.

Let us first fix some constants $c_1, c_2 > 0$ such that

$$c_1 c_2 < 1. \tag{6.1}$$

Set $\tilde{c}_2 \triangleq \frac{c_2 + c_1^{-1}}{2}$, so that $c_2 < \tilde{c}_2 < c_1^{-1}$. Furthermore, as in previous section, by a slight abuse of notations we replace $\Theta^j$, $j = 1, 2$ in (3.3) by $\theta_j$, $j = 1, 2$, and still denote them by $b_i$, $\sigma_i$, $f_i$, $i = 1, 2, 3$. In what follows all assumptions involving coefficients in (5.4) will be in the sense that they hold uniformly for all $\theta_j$, $j = 1, 2$.

Recall again that it is essential to have $\sigma_3 h \neq 1$. We shall establish the theory for the Cases (2-a)-(2-c). We also remark that the case $\sigma = \sigma(t, x, y)$ satisfies Case (2-a) with $c_1 = 0$. Our first result corresponds to Case (2-a) and Theorem 5.7.

**Theorem 6.1** *Suppose that Assumption 2.1 and (6.1) are in force, and assume that $|\sigma_3| \leq c_1$ and $|h| \leq c_2$. Then there exists a constant $\delta > 0$, which depends only on $c_1$, $c_2$, and the Lipschitz constants in Assumption 2.1, such that whenever $T \leq \delta$, it holds that*

*(i) the FBSDE (1.1) has a unique solution $\Theta \in \mathbb{L}^2$;*

*(ii) the ODEs in (3.12) have solutions $\overline{\mathbf{y}}, \underline{\mathbf{y}}$ such that*

$$-\tilde{c}_2 \leq \underline{\mathbf{y}}_t \leq \overline{\mathbf{y}}_t \leq \tilde{c}_2, \quad \forall\, t \in [0, T]; \tag{6.2}$$

*(iii) there exists a random field $u$ such that, for all $t \in [0, T]$, $Y_t = u(t, X_t)$ and*

$$\underline{\mathbf{y}}_t \leq \frac{u(t, x_1) - u(t, x_2)}{x_1 - x_2} \leq \overline{\mathbf{y}}_t, \quad \textit{for any } \; x_1 \neq x_2. \tag{6.3}$$

*Proof.* (i) follows directly from [1]. To see (ii), we notice that $\underline{F}$ and $\overline{F}$ are uniformly Lipschitz continuous in $y$ for $y \in [-\tilde{c}_2, \tilde{c}_2]$ and denote by $L$ the uniform Lipschitz constant. We assume that (i) holds for some $\delta > 0$. Modifying $\delta$ if necessary we may assume that

$$\Big[\int_0^\delta e^{Lt}dt\Big]\Big[\sup_{|y|\le \tilde{c}_2}\sup_{t\in[0,T]}[|\overline{F}(t,y)| + |\underline{F}(t,y)|]\Big] \le \tilde{c}_2 - c_2.$$

Now for any $T < \delta$, note that $\tilde{y}^1 \stackrel{\triangle}{=} -\tilde{c}_2$ and $\tilde{\mathbf{y}}^2 \stackrel{\triangle}{=} \tilde{c}_2$ satisfy the following ODEs:

$$\begin{aligned}
\tilde{\mathbf{y}}^1_t &= -\tilde{c}_2 + [\tilde{c}_2 - c_2] + \int_t^T [\underline{F}(s,\tilde{\mathbf{y}}^1_s) - \underline{F}(s,-\tilde{c}_2)]ds,\\
\tilde{\mathbf{y}}^2_t &= \tilde{c}_2 - [\tilde{c}_2 - c_2] + \int_t^T [\overline{F}(s,\tilde{\mathbf{y}}^1_s) - \overline{F}(s,-\tilde{c}_2)]ds.
\end{aligned}$$

Following the arguments in Theorem 5.7 we prove (ii).

It remains to prove (iii). Let $\delta > 0$ be small enough so that both (i) and (ii) hold. For any $(t,x)$, denote the (unique) solution to FBSDE (1.1) starting from $(t,x)$ by $\Theta^{t,x}$, and define a random filed $u(t,x) \stackrel{\triangle}{=} Y^{t,x}_t$. The uniqueness of the solution to FBSDE then leads to that $Y^{t,x}_s = u(s, X^{t,x}_s)$, for all $s \in [t,T]$, $\mathbb{P}$-a.s. In particular, denoting $\Theta_t = \Theta^{0,x}_t$, we have $Y_t = u(t,X_t)$, $t \in [0,T]$.

Now let $x_1 \neq x_2$ be given, and recall (3.1) and (4.1). Following the arguments in [1], for a smaller $\delta$ if necessary, one can easily see that $|\nabla Y_t| \le \tilde{c}_2|\nabla X_t|$. This also implies that

$$|\nabla X_t| \le \frac{1}{1 - c_1\tilde{c}_2}|\nabla X_t - \sigma_3\nabla Y_t|.$$

Applying Theorem 4.2 we see that $\nabla X \neq 0$ and $\hat{Y} \stackrel{\triangle}{=} \nabla Y/\nabla X$ satisfies the BSDE (3.6) and (3.8). Then (6.3) follows from Lemma 3.2. ∎

Our next result corresponds to Case (2-b) and Theorem 5.8.

**Theorem 6.2** *Suppose that Assumption 2.1 and (6.1) are in force, and assume that $\sigma_3$ and $h$ satisfy one of the conditions in (5.15)-(5.18). Then there exists a constant $\delta > 0$, depending only on $c_1$, $c_2$, and the Lipschitz constants in Assumption 2.1, such that when $T \le \delta$, all the results in Theorem 6.1 hold true, except for (6.2), where the bounded solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ should satisfy the corresponding property of $h$ in (5.15)-(5.18), with $c_2$ replaced by $\tilde{c}_2$.*

*Proof.* Without loss of generality, we prove the result only for the case (5.15). The other cases can be proved similarly.

We first assume (i) holds. Note that

$$c_2^{-1} \le h \le L,$$

where $L$ is the uniform Lipschitz constant in Assumption 2.1. By similar arguments as those in Theorem 6.1 (ii), for $\delta$ small enough one can easily show that the ODEs in (3.12) have solutions $\overline{\mathbf{y}}, \underline{\mathbf{y}}$ such that

$$\tilde{c}_2^{-1} \le \underline{\mathbf{y}}_t \le \overline{\mathbf{y}}_t \le 2L, \quad \text{for all} \quad t \in [0,T]. \tag{6.4}$$

This proves (ii). (iii) follows from (i) and similar arguments as those in Theorem 6.1 (iii).

So it remains to prove (i). Our main idea is to reverse the roles of forward and backward components and then apply Theorem 6.1. To this end, we consider a simple transformation: $\tilde{X} \triangleq Y$ and $\tilde{Y} \triangleq X$. In other words, we define the coordinate change:

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} \triangleq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and correspondingly,} \quad \tilde{z} \triangleq \sigma(t,x,y,z).$$

Note that, under (5.15), both functions $z \mapsto \sigma(t,x,y,z)$ and $x \mapsto g(x)$ are invertible, that is, there exist functions $\hat{\sigma}$ and $\hat{g}$ such that

$$\hat{\sigma}(t,x,y,\sigma(t,x,y,z)) = z, \quad \hat{g}(g(x)) = x. \tag{6.5}$$

Define

$$\begin{aligned} &\tilde{\sigma}(t,\tilde{\theta}) \triangleq \hat{\sigma}(t,\tilde{y},\tilde{x},\tilde{z}), \quad \tilde{g}(\tilde{x}) \triangleq \hat{g}(\tilde{x}); \\ &\tilde{b}(t,\tilde{\theta}) \triangleq -f(t,\tilde{y},\tilde{x},\tilde{\sigma}(t,\tilde{\theta})), \quad \tilde{f}(t,\tilde{\theta}) \triangleq -b(t,\tilde{y},\tilde{x},\tilde{\sigma}(t,\tilde{\theta})), \end{aligned}$$

and consider a new FBSDE:

$$\begin{cases} \tilde{X}_t = \tilde{x} + \displaystyle\int_0^t \tilde{b}(s,\tilde{\Theta}_s)ds + \int_0^t \tilde{\sigma}(s,\tilde{\Theta}_s)dB_s; \\ \tilde{Y}_t = \tilde{g}(\tilde{X}_T) + \displaystyle\int_t^T \tilde{f}(s,\tilde{\Theta}_s)ds - \int_t^T \tilde{Z}_s dB_s, \end{cases} \quad t \in [0,T], \tag{6.6}$$

We now show that FBSDE (6.6) satisfies the conditions in Theorem 6.1. First, by definition of inverse functions and by (5.15), we have

$$\hat{\sigma}_1 + \hat{\sigma}_3\sigma_1 = 0, \quad \hat{\sigma}_2 + \hat{\sigma}_3\sigma_2 = 0, \quad \hat{\sigma}_3\sigma_3 = 1, \quad \text{and} \quad \hat{h}h = 1,$$

where $\hat{\sigma}_i, \hat{h}$, and more notations below, are defined in the spirit of (3.3) for the functions $\hat{\sigma}, \hat{g}$. Note that

$$\tilde{\sigma}_3 = \hat{\sigma}_3 = (\sigma_3)^{-1}, \quad \tilde{h} = \hat{h} = h^{-1}.$$

This implies that, by (5.15),

$$L^{-1} \le \tilde{\sigma}_3 \le c_1, \quad L^{-1} \le \tilde{h} \le c_2, \tag{6.7}$$

Next, note that

$$\tilde{b}_1 = -f_2 - f_3\tilde{\sigma}_1 = -f_2 - f_3\hat{\sigma}_2 = -f_2 - f_3\sigma_2(\sigma_3)^{-1}.$$

We see that $|\tilde{b}_1| \le C$. Similarly $|\tilde{\varphi}_j| \le C$ for $\varphi = b, \sigma, f$ and $j = 1, 2, 3$. Moreover, note that

$$\begin{aligned}
&|\tilde{g}(0)| = |\tilde{g}(0) - \tilde{g}(g(0)| \le L|g(0)|;\\
&|\tilde{\sigma}(t,0,0,0)| = |\hat{\sigma}(t,0,0,0)| = |\hat{\sigma}(t,0,0,0) - \hat{\sigma}(t,0,0,\sigma(t,0,0,0))| \le C|\sigma(t,0,0,0)|;\\
&|\tilde{b}(t,0,0,0)| \le |f(t,0,0,0)| + C|\sigma(t,0,0,0)|, \quad |\tilde{f}(t,0,0,0)| \le |b(t,0,0,0)| + C|\sigma(t,0,0,0)|.
\end{aligned}$$

Thus (2.1) holds for FBSDE (6.6).

We can now apply Theorem 6.1 to conclude that for some $\delta > 0$, the FBSDE (6.6) admits a unique solution $\tilde{\Theta} \in \mathbb{L}^2$ for all $T \le \delta$, and $\tilde{Y}_t = \tilde{u}(t, \tilde{X}_t)$ for some decoupling random field $\tilde{u}$. Moreover, by (6.7) and modifying the arguments in Theorem 6.1 slightly, we see that $\tilde{u}$ satisfies

$$\frac{1}{2L} \le \frac{\tilde{u}(t,\tilde{x}_1) - \tilde{u}(t,\tilde{x}_2)}{\tilde{x}_1 - \tilde{x}_2} \le \tilde{c}_2.$$

Then $\tilde{u}(t,\tilde{x})$ has an inverse function $u(t,x)$ in terms of $x$. Now for any $x$, let $\tilde{x} \overset{\triangle}{=} u(0,x)$ and let $\tilde{\Theta}$ be the unique solution to FBSDE (6.6) with initial value $\tilde{X}_0 = \tilde{x}$. Then it is straightforward to check that

$$X_t \overset{\triangle}{=} \tilde{Y}_t, \quad Y_t \overset{\triangle}{=} \tilde{X}_t, \quad Z_t \overset{\triangle}{=} \tilde{\sigma}(t, \tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t)$$

satisfy FBSDE (1.1) with initial value $X_0 = x$.

Finally, note that $|\tilde{Z}| \le |\tilde{\sigma}(t,0,0,0)| + C[|\tilde{X}| + |\tilde{Y}| + |\tilde{Z}|]$, it is clear that $(X,Y,Z) \in \mathbb{L}^2$. The proof is complete now. ■

Our final result corresponds to Case (2-c) and Theorem 5.9.

**Theorem 6.3** *Suppose that Assumption 2.1 and (6.1) are in force, and assume that $\sigma_3$ and $h$ satisfy one of the conditions in (5.19)-(5.22). Then there exists a constant $\delta > 0$, depending only on $c_1$, $c_2$, and the Lipschitz constants in Assumption 2.1, such that when $T \le \delta$, all the results in Theorem 6.1 hold true, except for (6.2), where the bounded solutions $\overline{\mathbf{y}}$ and $\underline{\mathbf{y}}$ should satisfy the corresponding property of $h$ in (5.19)-(5.22), with $c_2$ replaced by $\tilde{c}_2$.*

*Proof.* Again we consider only the case (5.19), and the oher cases can be argued similarly. Following similar arguments as in Theorem 6.2, we shall only prove (i).

Slightly different from the proof of Theorem 6.2 we consider a slightly more complicated transformation: $(\tilde{x}, \tilde{y}, \tilde{z}) \overset{\triangle}{=} \Phi[\varepsilon](x,y,z)$, where

$$\begin{bmatrix}\tilde{x}\\ \tilde{y}\end{bmatrix} \overset{\triangle}{=} \begin{bmatrix}2\varepsilon & 1\\ \varepsilon & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}, \quad \tilde{z} \overset{\triangle}{=} \varepsilon\sigma(t,x,y,z) + z. \tag{6.8}$$

Note that

$$- L \leq \sigma_3 \leq c_1, \quad 0 \leq h \leq c_2. \tag{6.9}$$

By choosing $\varepsilon > 0$ small enough we see that the mappings

$$z \mapsto \tilde{z} = \varepsilon\sigma(t,x,y,z) + z \quad \text{and} \quad x \mapsto 2\varepsilon x + g(x)$$

are both strictly increasing and thus both are invertible. Denote the corresponding inverse functions by $\hat{\sigma}$ and $\hat{g}$, respectively. Namely,

$$\hat{\sigma}(t,x,y,\varepsilon\sigma(t,x,y,z)+z) = z, \quad \hat{g}(2\varepsilon x + g(x)) = x. \tag{6.10}$$

Furthermore, from (6.8) we can solve $(x,y) = (\frac{\tilde{x}-\tilde{y}}{\varepsilon}, 2\tilde{y}-\tilde{x})$, the inverse transformation of $\Phi[\varepsilon]$ is thus

$$(x,y,z) = \Psi[\varepsilon](\tilde{x},\tilde{y},\tilde{z}) \stackrel{\triangle}{=} \Big(\frac{\tilde{x}-\tilde{y}}{\varepsilon}, 2\tilde{y}-\tilde{x}, \hat{\sigma}\Big(t, \frac{\tilde{x}-\tilde{y}}{\varepsilon}, 2\tilde{y}-\tilde{x}, \tilde{z}\Big)\Big).$$

We now consider the FBSDE (6.6) with the following new coefficients:

$$\begin{array}{rcl} \tilde{b}(t,\tilde{x},\tilde{y},\tilde{z}) & = & 2\varepsilon b(t,\Psi[\varepsilon](\tilde{x},\tilde{y},\tilde{z})) - f(t,\Psi[\varepsilon](\tilde{x},\tilde{y},\tilde{z})), \\ \tilde{f}(t,\tilde{x},\tilde{y},\tilde{z}) & = & -\varepsilon b(t,\Psi[\varepsilon](\tilde{x},\tilde{y},\tilde{z})) + f(t,\Psi[\varepsilon](\tilde{x},\tilde{y},\tilde{z})) \\ \tilde{\sigma}(t,\tilde{x},\tilde{y},\tilde{z}) & = & 2\varepsilon\sigma(t,\Psi[\varepsilon](\tilde{x},\tilde{y},\tilde{z})) + \hat{\sigma}(t, \dfrac{\tilde{x}-\tilde{y}}{\varepsilon}, 2\tilde{y}-\tilde{x}, \tilde{z}); \\ \tilde{g}(\tilde{x}) & = & \varepsilon\hat{g}(\tilde{x}) + g(\hat{g}(\tilde{x})). \end{array} \tag{6.11}$$

Our idea is again to apply Theorem 6.1. Note that

$$\hat{\sigma}_3[\varepsilon\sigma_3 + 1] = 1, \quad \hat{h}[2\varepsilon + h] = 1.$$

Then,

$$\tilde{\sigma}_3 = 2\varepsilon\sigma_3\hat{\sigma}_3 + \hat{\sigma}_3 = \frac{2\varepsilon\sigma_3 + 1}{\varepsilon\sigma_3 + 1}, \quad \tilde{h} = \varepsilon\hat{h} + h\hat{h} = \frac{\varepsilon + h}{2\varepsilon + h}.$$

By (6.9) and for $\varepsilon > 0$ small enough, we have

$$0 < \frac{1 - 2L\varepsilon}{1 - \varepsilon L} \leq \tilde{\sigma}_3 \leq \frac{1 + 2c_1\varepsilon}{1 + c_1\varepsilon} \stackrel{\triangle}{=} \overline{c}_1; \quad 0 < \frac{1}{2} \leq \tilde{h} \leq \frac{\varepsilon + c_2}{2\varepsilon + c_2} \stackrel{\triangle}{=} \overline{c}_2. \tag{6.12}$$

Since $c_1c_2 < 1$, we obtain

$$\overline{c}_1\overline{c}_2 = \frac{1 + 2c_1\varepsilon}{1 + c_1\varepsilon} \cdot \frac{\varepsilon + c_2}{2\varepsilon + c_2} < 1. \tag{6.13}$$

Moreover, note that

$$\hat{\sigma}_1 + \varepsilon\hat{\sigma}_3\sigma_1 = 0, \quad \hat{\sigma}_2 + \varepsilon\hat{\sigma}_3\sigma_2 = 0.$$

Then

$$\hat{\sigma}_1 = \frac{-\varepsilon\sigma_1}{1+\varepsilon\sigma_3}, \quad \hat{\sigma}_2 = \frac{-\varepsilon\sigma_2}{1+\varepsilon\sigma_3}$$

are bounded, and therefore,

$$\tilde{b}_1 = 2\varepsilon\Big[b_1\varepsilon^{-1} - b_2 + b_3[\hat{\sigma}_1\varepsilon^{-1} - \hat{\sigma}_2]\Big] - \Big[f_1\varepsilon^{-1} - f_2 + f_3[\hat{\sigma}_1\varepsilon^{-1} - \hat{\sigma}_2]\Big]$$

is bounded. Similarly one can check that all other coefficients are all uniformly Lipschitz continuous and (2.1) also holds for FBSDE (1.1). Then we can apply Theorem 6.1, with $c_1$, $c_2$ being replaced by $\bar{c}_1$, $\bar{c}_2$ here, to conclude that (6.6) with coefficients given by (6.11) admits a unique solution $\tilde{\Theta} \in \mathbb{L}^2$, for $T \le \delta$ and $\delta$ small enough. Furthermore, by (6.12) and following similar arguments as in Theorem 6.1, it holds that $\tilde{Y}_t = \tilde{u}(t, \tilde{X}_t)$ for some decoupling random field $\tilde{u}$, which satisfies, for $\tilde{x}_1 \neq \tilde{x}_2$, and $\overline{c}_3 \stackrel{\triangle}{=} \frac{\overline{c}_1^{-1}+\overline{c}_2}{2}$,

$$\frac{1}{4} \le \frac{\tilde{u}(t,\tilde{x}_1) - \tilde{u}(t,\tilde{x}_2)}{\tilde{x}_1 - \tilde{x}_2} \le \overline{c}_3.$$

This then implies that $\tilde{x} \mapsto \tilde{u}(t,\tilde{x})$ has an inverse, denoted by $u(t,x)$.

Now for any $x$, let $\tilde{x} \stackrel{\triangle}{=} 2\varepsilon x + u(0,x)$ and $\tilde{\Theta}$ be the unique solution to FBSDE (6.6) starting from $\tilde{X}_0 = \tilde{x}$. Then one can easily check that $\Theta := \Psi[\varepsilon](\tilde{\Theta})$ satisfies all the requirement. ■

# 7 Synthetic Analysis

In this section we summarize all the results proved in the previous sections and give a synthetic analysis for the solvability of FBSDE (1.1) over an arbitrary duration $[0,T]$.

We first establish the theory in the case $\sigma = \sigma(t,x,y)$, and we remark that the work [28] is a special case of this result.

**Theorem 7.1** *Assume all the conditions in Theorem 5.6 hold, and let $\underline{\mathbf{y}}, \overline{\mathbf{y}}$ be the bounded solutions of ODEs (3.12). Then*

*(i) FBSDE (1.1) possesses a decoupling filed $u$ satisfying (6.3).*

*(ii) FBSDE (1.1) admits a unique solution $\Theta \in \mathbb{L}^2$, and there exists a constant $C > 0$, depending only on $T$, the Lipschitz constant in Assumption 2.1, and the bound of $\underline{\mathbf{y}}, \overline{\mathbf{y}}$, such that*

$$\|\Theta\|^2_{\mathbb{L}^2} \le C[|x|^2 + I_0^2]. \tag{7.1}$$

.

*Proof.* (i) First, applying Theorem 5.6, there exists a constant $c_3 > 0$ such that:

$$-c_3 \le \underline{\mathbf{y}} \le \overline{\mathbf{y}} \le c_3.$$

Notice that in this case $\sigma_3 = 0$ and thus $c_1 = 0$ in Theorem 6.1. Let $\delta > 0$ be the constant determined by $(0, c_3)$ in Theorem 6.1, and $0 = t_0 < \cdots < t_n = T$ be a partition of $[0, T]$ such that $t_i - t_{i-1} \le \delta$, $i = 1, \cdots, n$. We first consider FBSDE (1.1) on $[t_{n-1}, t_n]$. Since $\underline{\mathbf{y}} \le h \le \overline{\mathbf{y}}$, we see that the Lipschitz constant of the terminal condition $g$ is less than $c_3$, then by Theorem 6.1 there exists a random field $u(t, x)$ for $t \in [t_{n-1}, t_n]$ such that (6.3) holds for all $t \in [t_{n-1}, t_n]$. In particular, this implies that $c_3$ is a Lipschitz constant of $u(t_{n-1}, x)$. Next, consider FBSDE (1.1) on $[t_{n-2}, t_{n-1}]$ with terminal condition $u(t_{n-1}, x)$. Applying Theorem 6.1 on $[t_{n-2}, t_{n-1}]$ we find $u$ on $[t_{n-2}, t_{n-1}]$ such that (6.3) holds for $t \in [t_{n-2}, t_{n-1}]$. Repeating this procedure backwardly finitely many times, we extend the random field $u$ to the whole interval $[0, T]$. Clearly it is a decoupling field satisfying (6.3).

(ii) We first note that the above $n$ is fixed. Since $u$ is uniformly Lipschitz continuous in $x$, applying Theorem 6.1 on each interval $[t_i, t_{i+1}]$ with initial value $X_{t_i} = 0$, we see that there exists a constant $C$ such that

$$\mathbb{E}\{|u(t_i, 0)|^2\} = \mathbb{E}\{|Y^{t_i,0}_{t_i}|^2\} \le C\mathbb{E}\{|u(t_{i+1}, 0)|^2\} + CI_0^2.$$

Note that $u(t_n, 0) = g(0)$, we see that, for a larger $C$,

$$\max_{0 \le i \le n} \mathbb{E}\{|u(t_i, 0)|^2\} \le CI_0^2.$$

Next, by Theorem 2.3 FBSDE (1.1) admits a unique global solution $\Theta$. Applying Theorem 6.1 on each interval $[t_i, t_{i+1}]$ again we obtain

$$\mathbb{E}\Big\{\sup_{t_i \le t \le t_{i+1}} [|X_t|^2 + |Y_t|^2] + \int_{t_i}^{t_{i+1}} Z_t^2 dt\Big\} \le C\mathbb{E}\Big\{|X_{t_i}|^2 + |u(t_{i+1}, 0)|^2\Big\} + CI_0^2. \tag{7.2}$$

This implies that

$$\mathbb{E}\{|X_{t_{i+1}}|^2\} \le C\mathbb{E}\Big\{|X_{t_i}|^2 + |u(t_{i+1}, 0)|^2\Big\} + CI_0^2 \le C\mathbb{E}\{|X_{t_i}|^2\} + CI_0^2,$$

and thus

$$\max_{0 \le i \le n} \mathbb{E}\{|X_{t_i}|^2\} \le C[|x|^2 + I_0^2].$$

Plug this into (7.2) and sum over all $i$ we prove (7.1). ∎

For the general case $\sigma = \sigma(t, x, y, z)$, we assume that the standing Assumption 2.1 and (5.13) hold. In addition, we need one of the following assumptions: (5.14), (5.15)-(5.18), and (5.19)-(5.22). We now tabulate these conditions so that the nature of these assumptions are more

explicit, which in a sense could serve as a *User's Guide* for the solvability of the FBSDE (1.1). Let $\varepsilon > 0$ be given as that in Theorems 5.7, 5.8, 5.9, and $\alpha_3 \triangleq b_2 - \frac{b_3\sigma_2}{\sigma_3}$.

**Case I.** $|\sigma_3| \le c_1$, $|h| \le c_2$; and $\overline{F}(t, c_3) \le \varepsilon$, $\underline{F}(t, -c_3) \ge -\varepsilon$.

**Case II.** $|\sigma_3| \ge c_1^{-1}$, $|h| \ge c_2^{-1}$, and both of them keep the same sign.

| | $h \ge c_2^{-1}$ | $h \le -c_2^{-1}$ |
|---|---|---|
| $\sigma_3 \ge c_1^{-1}$ | $\underline{F}(t, c_3^{-1}) \ge -\varepsilon$, $\alpha_3 \le \varepsilon$ | $\overline{F}(t, c_3^{-1}) \le \varepsilon$, $\alpha_3 \ge -\varepsilon$ |
| $\sigma_3 \le -c_1^{-1}$ | $\underline{F}(t, c_3^{-1}) \ge -\varepsilon$, $\alpha_3 \le \varepsilon$ | $\overline{F}(t, c_3^{-1}) \le \varepsilon$, $\alpha_3 \ge -\varepsilon$ |

**Case III.** $\sigma_3 h \le c_1 c_2$, and one of them keeps the same sign.

| | $h \ge 0$ | $h \le 0$ | $\sigma_3 \ge 0$ | $\sigma_3 \le 0$ |
|---|---|---|---|---|
| $\sigma_3 \le c_1$, $h \le c_2$ | $\overline{F}(t, c_3) \le \varepsilon$, $f_1 \ge 0$ | | $\overline{F}(t, c_3) \le \varepsilon$ $\alpha_3 \ge -\varepsilon$ | |
| $\sigma_3 \ge -c_1$, $h \ge -c_2$ | | $\underline{F}(t, -c_3) \ge -\varepsilon$ $f_1 \le 0$ | | $\underline{F}(t, -c_3) \ge -\varepsilon$, $\alpha_3 \le \varepsilon$ |

Our main result is:

**Theorem 7.2** *Suppose that Assumption 2.1 and (5.13) are in force, and for $c_1$, $c_2$, $c_3$ in (5.13), either one of the conditions listed in Case I-III holds. Then*

*(i) FBSDE (1.1) possesses a decoupling filed $u$ such that $\frac{u(t,x_1)-u(t,x_2)}{x_1-x_2}$ satisfies the corresponding property of $h$ with $c_2$ replaced by $c_3$.*

*(ii) FBSDE (1.1) admits a unique solution $\Theta \in \mathbb{L}^2$, and there exists a constant $C > 0$, depending only on $T$, the Lipschitz constant in Assumption 2.1, and $c_1, c_2, c_3$, such that (7.1) holds.*

The proof is similar to that of Theorem 7.1 and is thus omitted. In particular, we emphasize that when one applies Theorem 6.1, 6.2, or 6.3, the constant $\delta$ should be determined by $c_1, c_3$, not by $c_1, c_2$!

The following special case deserves special attention.

**Corollary 7.3** *Let Assumption 2.1 hold. If, for arbitrary coefficients defined in (3.3) and (3.10),*

$$\sigma_3 \ge 0, \quad h \le 0, \quad f_1 \le 0, \quad b_2 - \frac{b_3\sigma_2}{\sigma_3} \ge 0; \tag{7.3}$$

*or*

$$\sigma_3 \le 0, \quad h \ge 0, \quad f_1 \ge 0, \quad b_2 - \frac{b_3\sigma_2}{\sigma_3} \le 0; \tag{7.4}$$

*then, for any $T$, FBSDE (1.1) is wellposed.*

*Proof.* We assume (7.3) holds. Set $c_1$ be the Lipschitz constant of $\sigma$ with respect to $z$, and let $0 < c_2 < c_3 < \delta$ for some $\delta$ small enough. One can easily check that (5.20) holds. ∎

**Comparison to the existing methods.** We now compare our conditions to several well-known existing results, and show that all of them are the special cases of our framework.

**1. Method of Contraction Mapping.** The original work Antonelli [1] assumes that $|\sigma_z g_x| < 1$ and that $T$ is small enough. This is covered by Theorem 6.1.

In the work Pardoux-Tang [16] it is essentially assumed, besides $\sigma_3$ and $h$ satisfy condition (5.14), that one of the following conditions holds:

(i) *"weak coupling":* That is, either $b_2, b_3, \sigma_2, \sigma_3$ are small or $f_1, h$ are small;

(ii) *"strong monotone":* That is, either $b_1$ is very negative or $f_2$ is very negative.

Now let us recall (3.7). For fixed $T$, the first condition implies that the coefficients of $y^2$ and $y^3$ is small enough and thus the ODEs (3.12) has desired solutions on $[0, T]$. The second condition implies that the coefficient of $y$ is very negative, which ensures that the solution to ODEs (3.12) will not blow up before $T$. ∎

**2. Method of Continuation.** The works Hu-Peng [6], Peng-Wu [17], Yong [23] relies heavily on the following "monotonicity condition":

$$\Delta b \Delta y + \Delta\sigma\Delta z - \Delta f \Delta x \ge \beta[|\Delta x|^2 + |\Delta y|^2 + |\Delta z|^2], \quad \Delta g \Delta x \le -\beta|\Delta x|^2, \tag{7.5}$$

for some constant $\beta > 0$. By some simple analysis, one sees immediately that

$$b_2 \ge \beta, \quad \sigma_3 \ge \beta, \quad f_1 \le -\beta \le 0, \quad h \le -\beta \le 0.$$

Moreover, by setting $\Delta x = 0$, we see that

$$b_2|\Delta y|^2 + \sigma_3|\Delta z|^2 + (b_3 + \sigma_2)\Delta y \Delta z \ge 0 \quad \text{for any} \quad \Delta y, \Delta z.$$

Then

$$(b_3 + \sigma_2)^2 - 4b_2\sigma_3 \le 0 \quad \text{and thus} \quad b_2\sigma_3 \ge \frac{1}{4}(b_3 + \sigma_2)^2 \ge b_3\sigma_2.$$

This implies (7.3) and thus the FBSDE is wellposed. We also noted that the monotone condition can be further weaken in our framework. ∎

**3. Four Step Scheme.** We should note that our solvability conditions (5.14), (5.15)-(5.18), (5.19)-(5.22) do not cover the results in [11] and [4]. However, in these cases by using the PDE arguments the deterministic decoupling function $u$ is uniformly Lipschitz continuous. ∎

# 8 Properties of the Solution

In this section we establish some further properties of the solution to the FBSDE (1.1). These will include a stability result, an $\mathbb{L}^p$-estimate for $p > 2$, and a comparison theorem for FBSDE.

We first prove the stability result.

**Theorem 8.1 (Stability)** *Assume both $(b, \sigma, f, g)$ and $(\tilde{b}, \tilde{\sigma}, \tilde{f}, \tilde{g})$ satisfy the same conditions (that is, they belong to the same case) in Theorem 7.2 (or Theorem 7.1). Let $u, \tilde{u}$ be the corresponding random fields and, for any $(t, x)$, $\Theta^{t,x}$ and $\tilde{\Theta}^{t,x}$ the solutions to the corresponding FBSDEs. For $\varphi = b, \sigma, f, g$, denote $\Delta\varphi \stackrel{\triangle}{=} \tilde{\varphi} - \varphi$. Then*

$$
\begin{aligned}
&\|\tilde{\Theta}^{0,\tilde{x}} - \Theta^{0,x}\|^2_{\mathbb{L}^2} &(8.1)\\
\leq\;& C\mathbb{E}\Big\{|\tilde{x} - x|^2 + |\Delta g(X^{0,x}_T)|^2 + \Big(\int_0^T [|\Delta b| + |\Delta f|](t, \Theta^{0,x}_t)dt\Big)^2 + \int_0^T |\Delta\sigma|^2(t, \Theta^{0,x}_t)dt\Big\},\\
&|\tilde{u}(t, x) - u(t, x)|^2 &(8.2)\\
\leq\;& C\mathbb{E}_t\Big\{|\Delta g(X^{t,x}_T)|^2 + \Big(\int_t^T [|\Delta b| + |\Delta f|](s, \Theta^{t,x}_s)ds\Big)^2 + \int_t^T |\Delta\sigma|^2(s, \Theta^{t,x}_s)ds\Big\}, \quad a.s.
\end{aligned}
$$

*Proof.* Note that $\tilde{u}(t, x) - u(t, x) = \tilde{Y}^{t,x}_t - Y^{t,x}_t$, and Consider the FBSDEs on $[t, T]$ and replace $\mathbb{E}$ with $\mathbb{E}_t$, (8.2) follows directly from (8.1).

To show (8.1), denote $\Delta\Theta \stackrel{\triangle}{=} \tilde{\Theta}^{0,\tilde{x}} - \Theta^{0,x}$ and $\Delta x \stackrel{\triangle}{=} \tilde{x} - x$. Then

$$
\begin{aligned}
\Delta X_t \;=\;& \Delta x + \int_0^t [\tilde{b}_1\Delta X_s + \tilde{b}_2\Delta Y_s + \tilde{b}_3\Delta Z_s + \Delta b(s, \Theta^{0,x}_s)]ds\\
&+ \int_0^t [\tilde{\sigma}_1\Delta X_s + \tilde{\sigma}_2\Delta Y_s + \tilde{\sigma}_3\Delta Z_s + \Delta\sigma(s, \Theta^{0,x}_s)]dB_s;\\
\Delta Y_t \;=\;& \tilde{h}\Delta X_T + \Delta g(X^{0,x}_T) + \int_t^T [\tilde{f}_1\Delta X_s + \tilde{f}_2\Delta Y_s + \tilde{f}_3\Delta Z_s + \Delta f(s, \Theta^{0,x}_s)]ds - \int_t^T \Delta Z_s dB_s.
\end{aligned}
$$

Here the notations $\tilde{b}_1$ etc are defined similar to (3.3). One can easily check that the above linear FBSDE (with solution $\Delta\Theta$) satisfies the corresponding conditions in Theorem 7.2 (or Theorem 7.1). Then applying the theorem we obtain the estimate immediately. ∎

We next establish the $L^p$-estimates for some $p > 2$. First, following Karatzas-Shreve [7] (Case 2 and 4, page 164), one can easily prove the following lemma.

**Lemma 8.2** *For any $p \geq 2$ and $Z \in L^{2,p}$, that is, $E\Big[\Big(\int_0^T |Z_t|^2 dt\Big)^{\frac{p}{2}}\Big] < \infty$, we have*

$$|\psi_1(p)|^{-p} E\Big[\Big|\int_0^t Z_s dB_s\Big|^p\Big] \leq E\Big[\Big(\int_0^t |Z_s|^2 ds\Big)^{\frac{p}{2}}\Big] \leq |\psi_2(p)|^p E\Big[\Big|\int_0^t Z_s dB_s\Big|^p\Big], \tag{8.3}$$

*where*

$$\psi_1(p) \stackrel{\triangle}{=} 2^{-\frac{1}{p}} p^{\frac{1}{2}} \Big(\frac{2^{\frac{p}{2}} - 2}{p-2}\Big)^{\frac{1}{2}-\frac{1}{p}}, \qquad \psi_2(p) \stackrel{\triangle}{=} \Big(\frac{p-1}{2}\Big)^{\frac{1}{p}} p^{\frac{1}{2}} \Big[2^{\frac{p}{2}} + \frac{2^{\frac{p}{2}} - 2}{p-2}\Big]^{\frac{1}{2}-\frac{1}{p}}. \tag{8.4}$$

*Moreover, for $i = 1, 2$, $\psi_i$ is continuous, strictly increasing on $[2, \infty)$ and $\psi_i(2) = 1$, $\psi_i(\infty) = \infty$.*

We now give the $L^p$-estimate of the solutions.

**Theorem 8.3 ($L^p$-estimates)** *Let $(b, \sigma, f, g)$ satisfy the conditions in Theorem 7.2. Assume*

$$2 \leq p < \psi^{-1}\Big(\frac{1}{c_1 c_3}\Big), \tag{8.5}$$

*where $\psi \stackrel{\triangle}{=} \psi_1 \psi_2$ and $\psi^{-1}$ denote the inverse function of $\psi$; and*

$$I_p^p \stackrel{\triangle}{=} \mathbb{E}\Big\{\Big(\int_0^T [|b| + |f|](t,0,0,0)dt\Big)^p + \Big(\int_0^T |\sigma|^2(t,0,0,0)dt\Big)^{\frac{p}{2}} + |g(0)|^p\Big\} < \infty. \tag{8.6}$$

*Then the unique solution $\Theta$ of FBSDE (1.1) is in $L^p$ and satisfies*

$$\|\Theta\|_{L^p} \leq C_p\Big[|x| + I_p\Big]. \tag{8.7}$$

*Consequently, the corresponding characteristic BSDE (3.6) has a unique solution $(\hat{Y}, \hat{Z})$ satisfying (3.8) and (4.14).*

*Proof.* By Theorem 7.2 and following its arguments, we may assume $p > 2$ and shall only prove the theorem under (5.14) and for $T \leq \delta$, where $\delta$ is a constant which depends on $c_1, c_3$, the Lipschitz constants, and $p$ and will be specified later. Moreover, by using the standard stopping arguments, we can assume without loss of generality that

$$\|\Theta\|_{w,p}^p \stackrel{\triangle}{=} \mathbb{E}\Big[\int_0^T [|X_t|^p + |Y_t|^p]dt + \Big(\int_0^T |Z_t|^2 dt\Big)^{\frac{p}{2}}\Big] < \infty. \tag{8.8}$$

For any $0 < \varepsilon \leq 1$ and $a, b > 0$, note that $(a+b)^p \leq (1 + C_p \varepsilon^{-1})a^p + (1+\varepsilon)b^p$. Then, for any $0 \leq t \leq T \leq \delta$, we have (denoting $\varphi_s = \varphi(s, \Theta_s)$, $\varphi = b, \sigma$, for simplicity)

$$\begin{aligned}
\mathbb{E}[|X_t|^p] &\leq (1 + C_p\varepsilon^{-1}))\mathbb{E}\Big[|x|^p + \Big(\int_0^t |b_s| ds\Big)^p\Big] + (1+\varepsilon)\mathbb{E}\Big[\Big|\int_0^t \sigma_s dB_s\Big|^p\Big] \\
&\leq C_p\varepsilon^{-1}\mathbb{E}\Big[|x|^p + \Big(\int_0^t |b_s| ds\Big)^p\Big] + (1+\varepsilon)\psi_1(p)^p \mathbb{E}\Big[\Big(\int_0^t |\sigma_s|^2 ds\Big)^{\frac{p}{2}}\Big],
\end{aligned}$$

where the second inequality thanks to Lemma 8.2. Note that

$$\begin{aligned}
\Big[\int_0^t |b_s|ds\Big]^p &\leq C_p\Big\{\int_0^T [|b(s,0)|+|X_s|+|Y_s|+|Z_s|]ds\Big\}^p \\
&\leq C_p\Big\{\Big[\int_0^T |b(s,0)|ds\Big]^p + T^{p-1}\int_0^T [|X_s|^p+|Y_s|^p]ds + T^{\frac{p}{2}}\Big[\int_0^T |Z_s|^2 ds\Big]^{\frac{p}{2}}\Big\}, \\
\Big[\int_0^t |\sigma_s|^2 ds\Big]^{\frac{p}{2}} &\leq \Big(\int_0^T \Big[(1+C\varepsilon^{-1})[|\sigma(s,0)|^2+|X_s|^2+|Y_s|^2] + (1+\varepsilon)c_1^2|Z_s|^2\Big]ds\Big)^{\frac{p}{2}} \\
&\leq (1+C_p\varepsilon^{-1})\Big\{\int_0^T (1+C\varepsilon^{-1})[|\sigma(s,0)|^2+|X_s|^2+|Y_s|^2]ds\Big\}^{\frac{p}{2}} \\
&\quad +(1+\varepsilon)\Big[\int_0^T (1+\varepsilon)c_1^2|Z_s|^2 ds\Big]^{\frac{p}{2}} \\
&\leq C_p\varepsilon^{-\frac{p}{2}-1}\Big\{\int_0^T [|\sigma(s,0)|^2+|X_s|^2+|Y_s|^2]ds\Big\}^{\frac{p}{2}} + (1+\varepsilon)^{\frac{p}{2}+1}c_1^p\Big[\int_0^T |Z_s|^2 ds\Big]^{\frac{p}{2}} \\
&\leq C_p\varepsilon^{-\frac{p}{2}-1}\Big\{\Big[\int_0^T |\sigma(s,0)|^2 ds\Big]^{\frac{p}{2}} + T^{\frac{p}{2}-1}\int_0^T [|X_s|^p+|Y_s|^p]ds\Big\} \\
&\quad +(1+\varepsilon)^{\frac{p}{2}+1}c_1^p\Big[\int_0^T |Z_s|^2 ds\Big]^{\frac{p}{2}}.
\end{aligned}$$

In the above, $\varphi(s,0) \overset{\triangle}{=} \varphi(s,0,0,0)$, for $\varphi = b,\sigma$, respectively. Then,

$$\begin{aligned}
\mathbb{E}[|X_t|^p] &\leq C_p\varepsilon^{-1}\Big[|x|^p + I_p^p + \delta^{\frac{p}{2}}\|\Theta\|_{w,p}^p\Big] \\
&\quad +(1+\varepsilon)\psi_1(p)^p\Big[C_p\varepsilon^{-\frac{p}{2}-1}[I_p^p + \delta^{\frac{p}{2}-1}\|\Theta\|_{w,p}^p] + (1+\varepsilon)^{\frac{p}{2}+1}c_1^p\|\Theta\|_{w,p}^p\Big] \\
&\leq C_p\varepsilon^{-\frac{p}{2}-1}\Big[|x|^p + I_p^p + \delta^{\frac{p}{2}-1}\|\Theta\|_{w,p}^p\Big] + (1+\varepsilon)^{\frac{p}{2}+2}\psi_1(p)^p c_1^p\|\Theta\|_{w,p}^p. \qquad (8.9)
\end{aligned}$$

Next, by Theorem 7.2 we have

$$|Y_t|^2 \leq C\mathbb{E}_t\Big[|X_t|^2 + |g(0)|^2 + \Big(\int_t^T [|b|+|f|](s,0)ds\Big)^2 + \int_t^T |\sigma(s,0)|^2 dt\Big].$$

This implies that

$$\mathbb{E}[|Y_t|^p] \leq C_p\mathbb{E}[|X_t|^p] + C_pI_p^p. \qquad (8.10)$$

In particular,

$$|Y_0|^p \leq C_p[|x|^p + I_p^p]. \qquad (8.11)$$

Moreover, following standard arguments

$$\begin{aligned}
\mathbb{E}\Big[\Big|\int_0^T Z_t dB_t\Big|^p\Big] &= \mathbb{E}\Big[\Big|g(X_T) - g(0) + g(0) - Y_0 + \int_0^T f(t,\Theta_t)dt\Big|^p\Big] \\
&\leq (1+\varepsilon)\mathbb{E}\Big[\Big|g(X_T)-g(0)\Big|^p\Big] + C_p\varepsilon^{-1}\mathbb{E}\Big[|g(0)|^p + |Y_0|^p + \Big|\int_0^T f(t,\Theta_t)dt\Big|^p\Big] \\
&\leq (1+\varepsilon)c_3^p\mathbb{E}\Big[|X_T|^p\Big] + C_p\varepsilon^{-1}\Big[|x|^p + I_p^p + \delta^{\frac{p}{2}}\|\Theta\|_{w,p}^p\Big].
\end{aligned}$$

Now by the second inequality in (8.3) and (8.9), we have

$$
\begin{aligned}
\mathbb{E}\Big[\Big(\int_0^T |Z_t|^2 dt\Big)^{\frac{p}{2}}\Big] &\le (1+\varepsilon)c_3^p|\psi_2(p)|^p\mathbb{E}\Big[|X_T|^p\Big] + C_p\varepsilon^{-1}\Big[|x|^p + I_p^p + \delta^{\frac{p}{2}}\|\Theta\|_{w,p}^p\Big] \qquad (8.12)\\
&\le (1+\varepsilon)^{\frac{p}{2}+3}[\psi(p)c_1c_3]^p\|\Theta\|_{w,p}^p + C_p\varepsilon^{-\frac{p}{2}-1}\Big[|x|^p + I_p^p + \delta^{\frac{p}{2}-1}\|\Theta\|_{w,p}^p\Big].
\end{aligned}
$$

Set $\varepsilon = 1$ in (8.9), and plug (8.9), (8.10), (8.12) into (8.8), we get

$$
\begin{aligned}
\|\Theta\|_{w,p}^p &\le \mathbb{E}\Big[\Big(\int_0^T |Z_t|^2 dt\Big)^{\frac{p}{2}}\Big] + \delta \sup_{0\le t\le T}\mathbb{E}[|X_t|^p + |Y_t|^p] \qquad (8.13)\\
&\le \Big[(1+\varepsilon)^{\frac{p}{2}+3}[\psi(p)c_1c_3]^p + C_p\varepsilon^{-\frac{p}{2}-1}\delta^{\frac{p}{2}-1} + C_p\delta\Big]\|\Theta\|_{w,p}^p + C_pC_p\varepsilon^{-\frac{p}{2}-1}[|x|^p + I_p^p].
\end{aligned}
$$

Denote

$$
c_p \overset{\triangle}{=} [\psi(p)c_1c_3]^p < 1.
$$

We may first choose $\varepsilon$ such that $(1+\varepsilon)^{\frac{p}{2}+3}[\psi(p)c_1c_3]^p = \frac{2c_p+1}{3}$, and then choose $\delta$ such that $C_p\varepsilon^{-\frac{p}{2}-1}\delta^{\frac{p}{2}-1} + C_p\delta = \frac{1-c_p}{6}$. Then (8.13) implies that

$$
\|\Theta\|_{w,p}^p \le \frac{c_p+1}{2}\|\Theta\|_{w,p}^p + C_p[|x|^p + I_p^p].
$$

Since $\frac{c_p+1}{2} < 1$, we obtain

$$
\|\Theta\|_{w,p}^p \le C_p[|x|^p + I_p^p].
$$

Now following standard arguments we can prove (8.7) straightforwardly.

Finally, the claim on $(\hat{Y}, \hat{Z})$ follows from Theorem 4.3 immediately. ■

We note that if $\sigma = \sigma(t,x,y)$, then we could simply take $c_1 = 0$. Note that $\psi^{-1}(\infty) = \infty$, by combining the arguments in Theorems 7.1 and 8.3 (noting (8.5)!) we obtain the following result immediately.

**Corollary 8.4** *Let $(b,\sigma,f,g)$ satisfy the conditions in Theorem 7.1. For any $p \ge 2$, if $I_p < \infty$, then the unique solution $\Theta$ of FBSDE (1.1) is in $L^p$ and satisfies (8.7). Consequently, the corresponding characteristic BSDE (3.6) has a unique solution $(\hat{Y}, \hat{Z})$ satisfying (3.8) and (4.14).*

For FBSDE (4.1), we have $I_p = 0$ for all $p \ge 2$, which leads to the following result.

**Corollary 8.5** *Assume the linear FBSDE (4.1) satisfy the conditions in Theorem 7.2 (or Theorem 7.1). Then any $2 \le p < \psi^{-1}(\frac{1}{c_1c_3})$, the unique solution $\Theta$ of FBSDE (4.1) is in $L^p$. Consequently, the corresponding characteristic BSDE (3.6) has a unique solution $(\hat{Y}, \hat{Z})$ satisfying (3.8) and (4.14).*

Finally, as an application of Corollary 8.5, we prove the comparison theorem.

**Theorem 8.6 (Comparison)** *Assume both $(b, \sigma, f, g)$ and $(b, \sigma, \tilde{f}, \tilde{g})$ satisfy the same conditions (that is, they belong to the same case) in Theorem 7.2 (or Theorem 7.1), and let $u, \tilde{u}$ be the corresponding random fields. If $f \le \tilde{f}, g \le \tilde{g}$, then $u \le \tilde{u}$.*

*Proof.* Without loss of generality, we shall prove the result only at $t = 0$. Let $\Theta, \tilde{\Theta} \in L^2$ be the corresponding solutions to the FBSDE (1.1) associated to $(b, \sigma, f, g)$ and $(b, \sigma, \tilde{f}, \tilde{g})$, respectively. Denote $\Delta\Theta_t \stackrel{\triangle}{=} \Theta_t - \tilde{\Theta}_t$, and define $\varphi_i$ similar to (3.3) for $\varphi = b,\ \sigma,\ f$, respectively. Then $\Delta\Theta$ would be the unique solution to the following linear FBSDE:

$$\begin{cases} \Delta X_t = \displaystyle\int_0^t (b_1 \Delta X_s + b_2 \Delta Y_s + b_3 \Delta Z_s) ds + \int_0^t (\sigma_1 \Delta X_s + \sigma_2 \Delta Y_s + \sigma_3 \Delta Z_s) dB_s; \\ \Delta Y_t = h \Delta X_T + \Delta g(\tilde{X}_T) + \displaystyle\int_t^T (f_1 \Delta X_s + f_2 \Delta Y_s + f_3 \Delta Z_s + \Delta f(t, \tilde{\Theta}_t)) ds - \int_t^T \Delta Z_s dB_s. \end{cases} \tag{8.14}$$

Let $(\hat{Y}, \hat{Z})$ denote the unique solution to BSDE (3.6) which, by Corollary 8.5, satisfies (3.8) and (4.14). Denote

$$\delta Y \stackrel{\triangle}{=} \Delta Y - \hat{Y} \Delta X, \quad \delta Z \stackrel{\triangle}{=} \Delta Z - \hat{Z} \Delta X - \hat{Y}[\sigma_1 \Delta X + \sigma_2 \Delta Y + \sigma_3 \Delta Z],$$

and define $\beta, \gamma$ and $\Gamma$ by (4.17) and (4.18). Applying Itô's formula we have

$$\delta Y_0 = \Gamma_0 \delta Y_0 = \Gamma_T \Delta g(\tilde{X}_T) + \int_0^T \Gamma_t \Delta f(t, \tilde{\Theta}_t) dt - \int_0^T \Gamma_t [\gamma_t \delta Y_t + \delta Z_t] dB_t.$$

Now by (4.14) and following similar arguments as in Theorem 4.3 one can easily show that $\int_0^t \Gamma_s [\gamma_s \delta Y_s + \delta Z_s] dB_s$ is a true martingale. Then by our assumptions we see that

$$u(0, x) - \tilde{u}(0, x) = \Delta Y_0 = \delta Y_0 = \mathbb{E}\Big\{\Gamma_T \Delta g(\tilde{X}_T) + \int_0^T \Gamma_t \Delta f(t, \tilde{\Theta}_t) dt\Big\} \le 0.$$

This proves the theorem. ■

**Remark 8.7** We notice that we cannot get $\Delta Y_t \ge 0$ even $\Gamma_t \ge 0$, $0 \le t \le T$, in the above proof process. This coincides with the results in Wu and Xu [21] (Theorem 3.2 and counter-example 3.1). However, for the corresponding random decoupling field, the comparison theorem holds over all time which coincides with Theorem 4.1 in Cvitanic and Ma [3] by virtue of PDE method under Markovian frame work. ■

# 9 Appendix

In this Appendix we complete the technical proofs for some results in Section 5.

**Proof of Lemma 5.1.** We first show the existence. Define a truncation function

$$\tilde{F}(t,y) \triangleq F\big(t, \mathbf{y}^1_t \vee y \wedge \mathbf{y}^2_t\big),$$

then by assumption (iii) $\tilde{F}$ is uniformly Lipschitz continuous in $y$ with a Lipschitz constant $L$, and thus the following ODE has a unique solution $\tilde{\mathbf{y}}$:

$$\tilde{\mathbf{y}}_t = h + \int_t^T \tilde{F}(s, \tilde{\mathbf{y}}_s)ds, \qquad t \in [0,T]. \tag{9.1}$$

We claim that

$$\mathbf{y}^1 \le \tilde{\mathbf{y}} \le \mathbf{y}^2. \tag{9.2}$$

This would lead to that $\tilde{F}(t,\tilde{\mathbf{y}}_t) = F(t,\tilde{\mathbf{y}}_t)$. Thus $\tilde{\mathbf{y}}$ is a solution to ODE (5.1) and (9.2) holds.

In fact, denote $\Delta\mathbf{y}^2 \triangleq \mathbf{y}^2 - \tilde{\mathbf{y}}$, $\Delta h^2 \triangleq h^2 - h$, $\Delta F^2 \triangleq F^2 - F$. Note that $F(t,\mathbf{y}^2_t) = \tilde{F}(t,\mathbf{y}^2_t)$, we have

$$\begin{aligned}
\Delta\mathbf{y}^2_t &= \Delta h^2 + C^2 + \int_t^T [F^2(s,\mathbf{y}^2_s) - \tilde{F}(s,\tilde{\mathbf{y}}_s) - c^2_s]ds \\
&= \Delta h^2 + C^2 + \int_t^T [\Delta F^2(s,\mathbf{y}^2_s) + \alpha_s \Delta\mathbf{y}^2_s - c^2_s]ds,
\end{aligned}$$

where $\alpha_s \triangleq \frac{\tilde{F}(s,\mathbf{y}^2_s) - \tilde{F}(s,\tilde{\mathbf{y}}_s)}{\Delta\mathbf{y}^2_s}\mathbf{1}_{\{\Delta\mathbf{y}^2_s \neq 0\}}$ satisfies $|\alpha| \le L$. Now define $\gamma_t \triangleq \exp(\int_0^t \alpha_s ds) > 0$. Then

$$\begin{aligned}
\gamma_t\Delta\mathbf{y}^2_t &= \gamma_T[\Delta h^2 + C^2] + \int_t^T \gamma_s[\Delta F^2(s,\mathbf{y}^2_s) - c^2_s]ds \\
&= \gamma_T \Delta h^2 + \int_t^T \gamma_s \Delta F^2(s,\mathbf{y}^2_s)ds + \gamma_T\Big[C^2 - \int_t^T \gamma_T^{-1}\gamma_s c^2_s ds\Big] \ge 0.
\end{aligned}$$

This implies that $\tilde{\mathbf{y}} \le \mathbf{y}^2$. Similarly we have $\tilde{\mathbf{y}} \ge \mathbf{y}^1$.

It remains to prove the uniqueness. Let $\mathbf{y}$ be an arbitrary solution to ODE (5.1) satisfying (9.2). Then $\tilde{F}(t,\mathbf{y}_t) = F(t,\mathbf{y}_t)$ and thus $\mathbf{y}$ satisfies ODE (9.1). By the uniqueness of ODE (9.1) we have $\mathbf{y} = \tilde{\mathbf{y}}$, and thus uniqueness follows. ■

**Proof of Theorem 5.3.** (*Necessity*) For simplicity let us rewrite (5.4) as

$$F(y) = f_1 + a_1 y + a_2 y^2 + a_3 y^3, \tag{9.3}$$

where $a_3 = \sigma_2 b_3$, $a_2 = b_2 + f_3\sigma_2 + b_3\sigma_1$, $a_1 = f_2 + b_1 + \sigma_1 f_3$.

We shall show that if none of (i)-(iii) holds, then the solution of ODE (5.3) will blow-up in finite time, which would complete the proof. To this end, we assume without loss of generality that $F(h) \geq 0$. Since (i) does not hold, we have $F(h) > 0$ and $F$ has no zero point in $[h, \infty)$. Note that if $a_3 < 0$ or if $a_3 = 0$ and $a_2 < 0$, then $\lim_{y\to\infty} F(y) = -\infty$, which together with $F(h) > 0$ will imply that $F$ has a zero point in $[h, \infty)$. Since (iii) does not hold either, we have

$$\text{either } a_3 > 0 \text{ or } a_3 = 0, a_2 > 0.$$

We investigate the two cases separately.

**Case 1.** Assume $a_3 > 0$. We claim that, there exist $\varepsilon > 0$ and $y_1 < h$ such that

$$F(y) \geq \varepsilon(y - y_1)^3, \quad \text{for all} \quad y \geq h. \tag{9.4}$$

Indeed, in this case $F(y)$ is a polynomial of degree 3, it must have at least one real zero point. By our assumption $F$ has no zero point after $h$, then all real zero points must be in $(-\infty, h)$. If there are three real zero points, we list them as $-\infty < y_1 \leq y_2 \leq y_3 < h$. Then for any $y \geq h$ one has

$$F(y) = a_3 \prod_{i=1}^{3} (y - y_i) \geq a_3(y - y_1)^3. \tag{9.5}$$

On the other hand, if $F$ has only one real zero point, denoted as $y_1$, then may write

$$F(y) = a_3(y - y_1)\Big[(y - y_2)^2 + c\Big] \quad \text{for some} \quad c > 0.$$

Note that the function $\tilde{F}(y) \stackrel{\triangle}{=} a_3[(y - y_2)^2 + c](y - y_1)^{-2}$ is continuous for $y \in [h, \infty)$, $\tilde{F}(y) > 0$ and $\lim_{y\to\infty} \tilde{F}(u) = a_3 > 0$. Then

$$\varepsilon \stackrel{\triangle}{=} \inf_{y\geq h} \frac{a_3[(y - y_2)^2 + c]}{(y - y_1)^2} > 0.$$

Thus, noting that $y - y_1 > 0$ for $y \geq h$,

$$F(y) = a_3(y - y_1)\Big[(y - y_2)^2 + c\Big] \geq \varepsilon(y - y_1)^3, \quad \text{for all} \quad y \geq h.$$

This, together with (9.5), proves (9.4).

Now consider the following ODE:

$$\tilde{\mathbf{y}}_t = h + \int_t^T \varepsilon(\tilde{\mathbf{y}}_t - y_1)^3 dt. \tag{9.6}$$

Solving this ODE we have

$$\tilde{\mathbf{y}}_t - y_1 = \frac{1}{\sqrt{2\varepsilon(t - T) + (h - y_1)^{-2}}}.$$

Thus, if $T > \frac{1}{2\varepsilon(h-y_1)^2}$, then the solution $\tilde{\mathbf{y}}_t$ blows up at $t = T - \frac{1}{2\varepsilon(h-y_1)^2} \in (0,T)$. On the other hand, by comparison theorem we can easily show that $\mathbf{y}_t \geq \tilde{\mathbf{y}}_t$. Thus the solution of (5.3) will blow-up at finite time as well.

**Case 2.** Assume $a_3 = 0$ and $a_2 > 0$. Following similar arguments, in this case we have

$$F(y) \geq \varepsilon(y-y_1)^2, \quad \text{for all} \quad y \geq h.$$

and similarly $\mathbf{y}$ will blow up if $T$ is large enough. ■

**Proof of Theorem 5.5.** (*Necessity*)

(i) Assume $h < \sigma_3^{-1}$, $F(h) \leq 0$, and $\alpha_3 \stackrel{\triangle}{=} b_2 - b_3\sigma_2\sigma_3^{-1} \neq 0$. We show that either $F$ has a zero in $(-\infty, h]$ or $\mathbf{y}$ blows up when $T$ is large enough.

Indeed, if $\alpha_3 > 0$, then $\lim_{y\to-\infty} F(y) = \infty$. Note that $F$ is continuous for $y \in (-\infty, h]$. These, together with $F(h) \leq 0$, imply that $F$ has a zero point in $(-\infty, h]$. We now assume $\alpha_3 < 0$. Denote $\tilde{F}(y) \stackrel{\triangle}{=} -\frac{F(y)}{(h+1-y)^2}$. In $(-\infty, h]$, if $F$ has no zero point, then $\tilde{F}$ is continuous, has no zero point, and $\lim_{y\to-\infty} \tilde{F}(y) = -\alpha_3 > 0$. Denote $\varepsilon \stackrel{\triangle}{=} \inf_{y\leq h} \tilde{F}(y) > 0$. Then we have

$$F(y) \leq -\varepsilon(h+1-y)^2 \quad \text{for all} \quad y \leq h.$$

Following the arguments for the proof of the necessary part of Theorem 5.3, we prove that $\mathbf{y}$ blows up when $T$ is large.

(ii) Assume $h > \sigma_3^{-1}$, $F(h) \geq 0$, and $\alpha_3 \neq 0$. Similarly we can show that either $F$ has a zero point in $[h, \infty)$ or $\mathbf{y}$ blows up when $T$ is large enough.

(iii) Assume $h < \sigma_3^{-1}$ and $F(h) \geq 0$. We show that either $F$ has a zero point in $[h, \sigma_3^{-1})$ or $\mathbf{y}$ violates (5.9) when $T$ is large enough.

Indeed, recall the $\alpha_0$ in (5.8). If $\alpha_0 < 0$, then $\lim_{y\uparrow\sigma_3^{-1}} F(y) = -\infty$. This implies that $F$ has a zero point in $[h, \sigma_3^{-1})$.

If $\alpha_0 > 0$ and $F$ has no zero point in $[h, \sigma_3^{-1})$. Denote $\tilde{F}(y) \stackrel{\triangle}{=} F(y)[\sigma_3^{-1} - y]$. Then in $[h, \sigma_3^{-1})$, $\tilde{F}$ is continuous, $\tilde{F} > 0$, and $\lim_{y\uparrow\sigma_3^{-1}} \tilde{F}(y) = \alpha_0 > 0$. Denote

$$\varepsilon \stackrel{\triangle}{=} \inf_{y\in[h,\sigma_3^{-1})} \tilde{F}(y) > 0, \quad \text{and thus} \quad F(y) \geq \varepsilon(\sigma_3^{-1} - y)^{-1} \quad \text{for } y \in [h, \sigma_3^{-1}).$$

Let $\tilde{\mathbf{y}}$ solve the following ODE:

$$\tilde{\mathbf{y}}_t = h + \int_t^T \varepsilon(\sigma_3^{-1} - \tilde{\mathbf{y}}_s)^{-1} ds,$$

we obtain explicitly

$$(\sigma_3^{-1} - \tilde{\mathbf{y}}_t)^2 = (\sigma_3^{-1} - h)^2 - 2\varepsilon(T - t).$$

Let $T \geq \frac{1}{2\varepsilon}(\sigma_3^{-1} - h)^2$. Then for $t = T - \frac{1}{2\varepsilon}(\sigma_3^{-1} - h)^2 \in [0, T]$, we have $\tilde{\mathbf{y}}_t = \sigma_3^{-1}$. By comparison we see that $(1 - \sigma_3 \mathbf{y})^{-1}$ would blow up.

Finally, if $\alpha_0 = 0$ and $F$ has no zero point in $[h, \sigma_3^{-1})$. Then $F$ is continuous and positive on $[h, \sigma_3^{-1}]$. Denote $\varepsilon \stackrel{\triangle}{=} \inf_{y \in [h, \sigma_3^{-1}]} F(y) > 0$, and define

$$\tilde{\mathbf{y}}_t \stackrel{\triangle}{=} h + \int_t^T \varepsilon ds = h + \varepsilon(T - t), \quad t \in [0, T].$$

Thus, if $T \geq \varepsilon^{-1}[\sigma_3^{-1} - h]$, then $\tilde{\mathbf{y}}_t = \sigma_3^{-1}$ at $t = T - \varepsilon^{-1}[\sigma_3^{-1} - h]$. By comparison again we see that $(1 - \sigma_3 \mathbf{y})^{-1}$ would blow up.

(iv) Assume $h > \sigma_3^{-1}$ and $F(h) \leq 0$. We can similarly show that either $F$ has a zero point in $(\sigma_3^{-1}, h]$ or $\mathbf{y}$ violates (5.9) when $T$ is large enough. ■

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